A fractional step θ-method for viscoelastic fluid flow using a SUPG approximation

J.C. Chrispell * V.J. Ervin † E.W. Jenkins *

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Abstract

In this article a fractional step θ-method is described and studied for the approximation of time dependent viscoelastic fluid flow equations, using the Johnson-Segalman constitutive model. The θ-method implementation allows the velocity and pressure approximations to be decoupled from the stress, reducing the number of unknowns resolved at each step of the method. The constitutive equation is stabilized using a Streamline Upwinded Petrov-Galerkin (SUPG)-method. A priori error estimates are given for the approximation scheme. Numerical computations supporting the theoretical results and demonstrating the θ-method are also presented.

Key words. θ-method; splitting method; viscoelastic flow

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1 Introduction

Numerical methods for modeling viscoelastic fluid flow are difficult for a variety of reasons. The modeling equations (assuming *slow* flow) represent a "Stokes system" for the conservation of mass and momentum equations, coupled with a non-linear hyperbolic equation describing the constitutive equation for the stress. The numerical approximation requires the determination of the fluid's velocity, pressure and stress (a symmetric tensor). For an accurate approximation a direct approximation technique requires the solution of a very large non-linear system of equations at each time step.

The fractional step θ-method [25, 23, 24] decouples the approximation of velocity and pressure from the approximation of the stress, thereby reducing the size of the algebraic systems which have to be solved at each sub-step. An added benefit of the θ-method [25] is that the algebraic systems to be solved at each sub-step are linear.

*Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA. This work was supported in part by the National Science Foundation under Award Number DMS-0410792, and U.S. Army Research Office Grant W911NF-05-1-0380. E-mail addresses: jchrisp@clemson.edu (J.C. Chrispell), lea@clemson.edu (E.W. Jenkins).

†Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA. This work was supported in part by the National Science Foundation under Award Number DMS-0410792. Corresponding author. E-mail address: vjervin@clemson.edu (V.J. Ervin)

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The fractional step θ-method was introduced, and its temporal approximation accuracy studied for a symmetric, positive definite spatial operator, by Glowinski and Pirreneau in [11]. The method is widely used for the accurate approximation of the Navier-Stokes equations (NSE) [26, 27, 14]. In [15], Kloucek and Rys showed, assuming a unique solution existed, that the θ-method approximation converged to the solution of the NSE as the spatial and mesh parameters went to zero ($h,\ \Delta t\ \rightarrow\ 0_{+}$). The temporal discretization error for the θ-method for the NSE was studied by Müller-Urbaniak in [18] and shown to be second order. In [7] the θ-method applied to convection-diffusion equations was shown to be second order in time.

The implementation of the fractional step θ-method in [25] for viscoelasticity differs significantly from that for the NSE. For the NSE at each sub-step the discretization contains the *stabilizing* operator $-\Delta\mathbf{u}$. For the viscoelasticity problem the middle-substep is a pure convection (transport) problem that requires stabilization in order to control the creation of spurious oscillations in the numerical approximation. Marchal and Crochet [17] were the first to use streamline upwinding to stabilize the hyperbolic constitutive equation in viscoelastic flow. A second common approach to stabilizing the convective transport problem is to use a discontinuous Galerkin (DG) approximation for the stress [2, 1].

Error analysis of finite element approximations to steady state viscoelastic flow was first done by Baranger and Sandri in [2] using a DG formulation of the constitutive equation. In [22] Sandri presented analysis of the steady state problem using a streamline upwind Petrov-Galerkin (SUPG) method of stabilization. The time-dependent problem was first analyzed by Baranger and Wardi in [3], using an implict Euler temporal discretization, and DG approximation for the hyperbolic constitutive equation. Ervin and Miles analyzed the problem using an implicit Euler time discretization, and a SUPG discretization for the stress in [9]. Analysis of a modified Euler-SUPG approximation to the transient viscoelastic flow problem was presented by Bensaada and Esselaoui in [4]. The temporal accuracy of the approximation schemes studied in [3, 4, 9] are all $O(\Delta t)$. Ervin and Heuer proposed a Crank-Nicolson time discretization method [8] which they showed was $O\left((\Delta t^2)\right)$. Their method uses a three level scheme to approximate the non-linear terms in the equations. Consequently their approximation algorithm only requires linear systems of equations to be solved.

In this article we analyze a fractional step θ-method for the approximation of viscoelastic fluid flows. Advantages of the method are three fold: (i) second order accuracy with respect to the temporal discretization, (ii) only linear systems of equations need to be solved, (iii) the linear systems to be solved only involve the velocity-pressure or the stress unknowns (resulting in smaller linear systems).

This paper is organized as follows. In the next section we specify the problem and describe the fractional step θ-method for the viscoelastic modeling equations. In Section 3 the mathematical notation used is given. In Section 4 we show computability of the algorithm, and present the a priori error estimates that support the method. Two numerical examples demonstrating the method are presented in Section 7.

2 The Mathematical Model and θ-Method Approximation

In this section we present the modeling equations for viscoelastic fluid flow as well as a fractional step θ-method approximation scheme. Following the description of the θ-method, several definitions

used to formulate the problem in an appropriate mathematical setting are given.

2.1 Modeling Equations

The non-dimensional modeling equations for a viscoelastic fluid in a given domain $\Omega \subset \mathbb{R}^{\acute{d}}$ ($\acute{d} = 2, 3$) using a Johnson-Segalman constitutive equation are written as:

$$
\begin{aligned}
\boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) \right) - 2\alpha \mathbf{d}(\boldsymbol{u}) &= \mathbf{0} \quad \text{in } \Omega, && (2.1)\\
Re \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) + \nabla p - 2(1-\alpha) \nabla \cdot \mathbf{d}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \quad \text{in } \Omega, && (2.2)\\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega, && (2.3)\\
\boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega, && (2.4)\\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0(x) \text{ in } \Omega, && (2.5)\\
\boldsymbol{\sigma}(0, x) &= \boldsymbol{\sigma}_0(x) \text{ in } \Omega. && (2.6)
\end{aligned}
$$

Here (2.1) is the constitutive equation relating the fluids velocity $\boldsymbol{u}$ to the stress $\boldsymbol{\sigma}$, and (2.2) and (2.3) are the conservation of momentum and conservation of mass equations. The fluid pressure is denoted by p. The Weissenberg number λ is a dimensionless constant defined as the product of a characteristic strain rate and the relaxation time of the fluid [5]. Re denotes the fluids Reynolds number, $\mathbf{f}$ are the body forces acting on the fluid, and $\alpha \in (0, 1)$ denotes the fraction of the total viscosity that is viscoelastic.

The term g_a and deformation tensor $\mathbf{d}(\boldsymbol{u})$ are defined as:

$$
g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) := \frac{1-a}{2} \left(\boldsymbol{\sigma} \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \boldsymbol{\sigma} \right) - \frac{1+a}{2} \left(\nabla \boldsymbol{u} \boldsymbol{\sigma} + \boldsymbol{\sigma} (\nabla \boldsymbol{u})^T \right)
$$

and

$$
\mathbf{d}(\boldsymbol{u}) = \frac{1}{2} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right).
$$

The gradient of $\boldsymbol{u}$ is defined such that $(\nabla \boldsymbol{u})_{i,j} = \partial u_i / \partial x_j$. For the remainder of the paper slow or inertialess flow is assumed, allowing the term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in (2.2) to be ignored. Note that when the parameter $a = 1$ in $g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u})$ an Oldroyd B constitutive model is obtained. For existence and uniqueness of solutions to (2.1)-(2.6) see [21, 12, 10].

2.2 θ-method

In order to implement a fractional step θ-method for the viscoelastic flow equations, an additive decomposition is used for equations (2.1) and (2.2). Here we introduce a splitting parameter $\omega \in (0, 1)$, and define:

Constitutive equation:

$$
\begin{aligned}
{}_1G_{\boldsymbol{\sigma}} &:= \omega \boldsymbol{\sigma}, && (2.7)\\
{}_2G_{\boldsymbol{\sigma}} &:= (1-\omega)\boldsymbol{\sigma} + \lambda \left(\boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) \right) - 2\alpha \mathbf{d}(\boldsymbol{u}). && (2.8)
\end{aligned}
$$

Conservation of Momentum:

$$
\begin{aligned}
{}_1F\boldsymbol{u} &:= -2(1-\alpha)\nabla\cdot\mathbf{d}(\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} - \mathbf{f}, && (2.9)\\
{}_2F\boldsymbol{u} &:= 0. && (2.10)
\end{aligned}
$$

Let Δt denote the temporal increment between times t^n and t^{n+1}, and for $c \in \{\theta, \omega, a, \alpha\}$ let $\tilde{c} := 1 - c$. Also, let $f^{(n)} := f(\cdot, n\Delta t)$.

The θ-method approximation for viscoelasticity may then be described as follows. (See also [25, 23]).

θ-Method Algorithm for Viscoelasticity

Step 1a: (Update the stress.)

$$
\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta\Delta t} + {}_1G\boldsymbol{\sigma}^{(n+\theta)} = -{}_2G\boldsymbol{\sigma}^{(n)}.
$$

Step 1b: (Solve for velocity and pressure.)

$$
\begin{aligned}
Re\frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta\Delta t} + \nabla p^{(n+\theta)} + {}_1F\boldsymbol{u}^{(n+\theta)} &= -{}_2F\boldsymbol{u}^{(n)},\\
\nabla\cdot\boldsymbol{u}^{(n+\theta)} &= 0.
\end{aligned}
$$

Step 2a: (Update the velocity and pressure.)

$$
\begin{aligned}
Re\frac{\boldsymbol{u}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\theta)}}{(1-2\theta)\,\Delta t} + \nabla p^{(n+\tilde{\theta})} + {}_2F\boldsymbol{u}^{(n+\tilde{\theta})} &= -{}_1F\boldsymbol{u}^{(n+\theta)},\\
\nabla\cdot\boldsymbol{u}^{(n+\tilde{\theta})} &= 0.
\end{aligned}
$$

Step 2b: (Solve for the stress.)

$$
\lambda \frac{\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\theta)}}{(1-2\theta)\,\Delta t} + {}_2G\boldsymbol{\sigma}^{(n+\tilde{\theta})} = -{}_1G\boldsymbol{\sigma}^{(n+\theta)}.
$$

Step 3a and Step 3b: In order to complete the temporal advancement to time t_{n+1}, Step 1a, and Step 1b are repeated with (n) and $(n+\theta)$ replaced by $\left(n+\tilde{\theta}\right)$ and $(n+1)$ respectively.

Note: Due to the chosen decomposition of the constitutive and conservation of momentum equations, (2.7)-(2.10), the approximation of the nonlinear system (2.1)-(2.6) using the θ-method only requires linear systems of equations to be solved at each step in the process.

3 Mathematical Notation

The $L^2(\Omega)$ inner product and norm are denoted by $(\cdot,\cdot)$, and $\|u\|$ respectively. The Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{W_p^k}$. We use H^k to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ to denote

the norm in H^k. Function spaces used in the analysis are:

$$\begin{aligned}
X &:= H_0^1(\Omega) := \left\{\boldsymbol{u} \in H^1(\Omega) : \boldsymbol{u} = 0 \text{ on } \partial\Omega\right\}, \\
S &:= \left\{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji}; \sigma_{ij} \in L^2(\Omega); 1 \le i,j \le \acute{d}\right\} \\
&\qquad \cap \left\{\boldsymbol{\sigma} = (\sigma_{ij}) : \boldsymbol{u} \cdot \nabla\boldsymbol{\sigma} \in L^2(\Omega),\ \forall \boldsymbol{u} \in X\right\}, \\
Q &:= L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\right\}, \\
Z &:= \left\{\mathbf{v} \in X : \int_\Omega q(\nabla \cdot \mathbf{v})\, dx = 0,\ \forall\, q \in Q\right\}.
\end{aligned}$$

Recall that the spaces X and Q satisfy the *inf-sup* condition

$$\inf_{q \in Q} \sup_{\mathbf{v} \in X} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\|\,\|\mathbf{v}\|_1} \ge \beta > 0. \tag{3.1}$$

A variational formulation of (2.1)-(2.3), found by multiplication of the modeling equations by test functions and integrating over Ω, is: *Find* $(\boldsymbol{u}, \boldsymbol{\sigma}, p) : (0,T] \to X \times S \times Q$ *such that*

$$\begin{aligned}
\lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{\tau}\right) + (\boldsymbol{\sigma}, \boldsymbol{\tau}) - 2\alpha\left(\mathbf{d}(\boldsymbol{u}), \boldsymbol{\tau}\right) + \lambda\left(\boldsymbol{u} \cdot \nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u}), \boldsymbol{\tau}\right) &= 0, \quad \forall\, \boldsymbol{\tau} \in S, && (3.2)\\
Re\left(\frac{\partial \boldsymbol{u}}{\partial t}, \mathbf{v}\right) - (p, \nabla \cdot \mathbf{v}) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}), \mathbf{d}(\mathbf{v})\right) + (\boldsymbol{\sigma}, \mathbf{d}(\mathbf{v})) &= (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in X, && (3.3)\\
(\nabla \cdot \boldsymbol{u}, q) &= 0, \quad \forall\, q \in Q, && (3.4)\\
\boldsymbol{u}(0,x) &= \boldsymbol{u}_0(x), && (3.5)\\
\boldsymbol{\sigma}(0,x) &= \boldsymbol{\sigma}_0(x). && (3.6)
\end{aligned}$$

As the velocity and pressure spaces X and Q satisfy the inf-sup condition (3.1), an equivalent variational formulation to (3.2)-(3.4) is given by: *Find* $(\boldsymbol{u}, \boldsymbol{\sigma}) : (0,T] \to Z \times S$ *such that*

$$\begin{aligned}
\lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{\tau}\right) + (\boldsymbol{\sigma}, \boldsymbol{\tau}) - 2\alpha\left(\mathbf{d}(\boldsymbol{u}), \boldsymbol{\tau}\right) + \lambda\left(\boldsymbol{u} \cdot \nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u}), \boldsymbol{\tau}\right) &= 0, \quad \forall\, \boldsymbol{\tau} \in S, && (3.7)\\
Re\left(\frac{\partial \boldsymbol{u}}{\partial t}, \mathbf{v}\right) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}), \mathbf{d}(\mathbf{v})\right) + (\boldsymbol{\sigma}, \mathbf{d}(\mathbf{v})) &= (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in Z, && (3.8)\\
\boldsymbol{u}(0,x) &= \boldsymbol{u}_0(x), && (3.9)\\
\boldsymbol{\sigma}(0,x) &= \boldsymbol{\sigma}_0(x). && (3.10)
\end{aligned}$$

To describe the finite element framework let T_h be a triangulation of the discretized domain $\Omega \subset \mathbb{R}^{\acute{d}}$. Then

$$\bar{\Omega} = \cup K, \quad K \in T_h.$$

We assume that there exist constants c_1 and c_2 such that

$$c_1 h \le h_K \le c_2 \rho_K,$$

where h_k is the diameter of triangle K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{k \in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials on A of degree no greater

than k and $C(\bar{\Omega})^{\acute{d}}$ the space of vector valued functions with $\acute{d}$ components which are continuous on $\bar{\Omega}$. Then the associated finite element spaces are defined by:

$$\begin{aligned}
X_h &:= \left\{\mathbf{v} \in X \cap C(\bar{\Omega})^{\acute{d}} : \mathbf{v}\big|_K \in P_k(K)\,\forall\, K \in T_h\right\},\\
S_h &:= \left\{\boldsymbol{\tau} \in S \cap C(\bar{\Omega})^{\acute{d}\times\acute{d}} : \boldsymbol{\tau}\big|_K \in P_m(K)\,\forall\, K \in T_h\right\},\\
Q_h &:= \left\{q \in Q \cap C(\bar{\Omega}) : q\big|_K \in P_q(K)\,\forall\, K \in T_h\right\},\\
Z_h &:= \left\{\mathbf{v} \in X_h : (q, \nabla\cdot\mathbf{v}) = 0\;\forall\, q \in Q_h\right\}.
\end{aligned}$$

Analogically to the continuous spaces assume that X_h and Q_h satisfy the discrete *inf-sup* condition:

$$\inf_{q\in Q_h}\sup_{\mathbf{v}\in X_h}\frac{(q,\nabla\cdot\mathbf{v})}{\|q\|\,\|\mathbf{v}\|_1} \geq \beta > 0. \tag{3.11}$$

Let $\mathcal{U}$ and $\mathcal{S}$ denote the L^2 projections of $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ onto Z_h and S_h, respectively, and define:

$$\begin{aligned}
\boldsymbol{\Lambda}^{(n)} &= \boldsymbol{u}^{(n)} - \mathcal{U}^{(n)}, & \boldsymbol{E}^{(n)} &= \mathcal{U}^{(n)} - \boldsymbol{u}_h^{(n)},\\
\boldsymbol{\Gamma}^{(n)} &= \boldsymbol{\sigma}^{(n)} - \mathcal{S}^{(n)}, & \boldsymbol{F}^{(n)} &= \mathcal{S}^{(n)} - \boldsymbol{\sigma}_h^{(n)},\\
e_{\boldsymbol{u}}^{(n)} &= \boldsymbol{u}^{(n)} - \boldsymbol{u}_h^{(n)}, & e_{\boldsymbol{\sigma}}^{(n)} &= \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}_h^{(n)}.
\end{aligned}$$

We define the discrete temporal operator

$$d_t f^{(n+1)} := \frac{f(t_{n+1}) - f(t_n)}{\Delta t}.$$

When $v(\mathbf{x}, t)$ is defined on the entire time interval $(0, T)$,

$$\|v\|_{\infty,k} := \sup_{0<t<T}\|v(\cdot,t)\|_k\,, \quad \|v\|_{0,k} := \left(\int_0^T \|v(\cdot,t)\|_k^2\,dt\right)^{1/2}, \quad \|v\|(t) := \|v(\cdot,t)\|\,.$$

The following discrete norms are used:

$$|||v|||_{\infty,k} := \max_{1\leq n\leq N}\left\|v^{(n)}\right\|_k, \qquad |||v|||_{0,k} := \left(\sum_{n=1}^{N}\Delta t\left\|v^{(n)}\right\|_k^2\right)^{\frac{1}{2}}.$$

4 Analysis

In this section we investigate the numerical approximation method corresponding to (3.7)-(3.10) First the discrete variational formulation of the θ-method is given. Then computability of the algorithm is shown and a priori error estimates given.

4.1 Discrete Variational Approximation

To stabilize the constitutive equation a streamline upwind Petrov-Galerkin (SUPG) discretization is used to control spurious oscillations in the approximation. This is implemented by testing all

terms in the constitutive equation (except the discretized temporal derivative) against modified test elements of the form $\boldsymbol{\tau}_{\delta^{(n)}}$ where

$$\boldsymbol{\tau}_{\delta^{(n)}} := \boldsymbol{\tau} + \delta \boldsymbol{u}_h^{(n)} \cdot \nabla \boldsymbol{\tau}. \tag{4.1}$$

Note that if δ is set to zero the standard Galerkin method is obtained. The variational formulations for the steps in the θ-method approximation are as follows.

Step 1a: Find $\boldsymbol{\sigma}_h^{(n+\theta)} \in S_h$ such that

$$\begin{aligned}\frac{\lambda}{\theta\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) + \omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) = \frac{\lambda}{\theta\Delta t}\left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}\right) - (1-\omega)\left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) - \lambda\left(\boldsymbol{u}_h^{(n)} \cdot \nabla\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) \\ - \lambda\left(g_a(\boldsymbol{\sigma}_h^{(n)}, \nabla\boldsymbol{u}_h^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}}\right) + 2\alpha\left(\mathbf{d}(\boldsymbol{u}_h^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}}\right), \qquad \forall\, \boldsymbol{\tau} \in S_h.\end{aligned} \tag{4.2}$$

Step 1b: Find $\boldsymbol{u}_h^{(n+\theta)} \in Z_h$ such that

$$\begin{aligned}\frac{Re}{\theta\Delta t}\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right) \\ = \frac{Re}{\theta\Delta t}\left(\boldsymbol{u}_h^{(n)}, \mathbf{v}\right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right), \qquad \forall\, \mathbf{v} \in Z_h.\end{aligned} \tag{4.3}$$

Step 2a: Find $\boldsymbol{u}_h^{(n+\tilde{\theta})} \in Z_h$ such that

$$\begin{aligned}\frac{Re}{(1-2\theta)\Delta t}\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \mathbf{v}\right) = \frac{Re}{(1-2\theta)\Delta t}\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) \\ - 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right), \qquad \forall\, \mathbf{v} \in Z_h.\end{aligned} \tag{4.4}$$

Step 2b: Find $\boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \in S_h$ such that

$$\begin{aligned}\frac{\lambda}{(1-2\theta)\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}\right) + (1-\omega)\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) + \lambda\left(\boldsymbol{u}_h^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\ + \lambda\left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) - 2\alpha\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\ = \frac{\lambda}{(1-2\theta)\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) - \omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right), \qquad \forall\, \boldsymbol{\tau} \in S_h.\end{aligned} \tag{4.5}$$

Step 3a: Find $\boldsymbol{\sigma}_h^{(n+1)} \in S_h$ such that

$$\begin{aligned}\frac{\lambda}{\theta\Delta t}\left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau}\right) + \omega\left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) = \frac{\lambda}{\theta\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}\right) - (1-\omega)\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\ - \lambda\left(\boldsymbol{u}_h^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, -\boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \lambda\left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla - \boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\ + 2\alpha\left(-\mathbf{d}(\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right), \qquad \forall\, \boldsymbol{\tau} - \in S_h.\end{aligned} \tag{4.6}$$

Step 3b: Find $\boldsymbol{u}_h^{(n+1)} \in Z_h$ such that

$$\frac{Re}{\theta \Delta t}\left(\boldsymbol{u}_h^{(n+1)}, \mathbf{v}\right) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+1)}), \mathbf{d}(\mathbf{v})\right) = \frac{Re}{\theta \Delta t}\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \mathbf{v}\right) + \left(\mathbf{f}^{(n+1)}, \mathbf{v}\right) - \left(\boldsymbol{\sigma}_h^{(n+1)}, \mathbf{d}(\mathbf{v})\right), \qquad \forall\, \mathbf{v} \in Z_h. \tag{4.7}$$

4.2 Existence and Uniqueness

Before error estimates are presented the computability of the θ-method is shown. This is accomplished by proving that the associated coefficient matrices used in each step of the algorithm are invertible.

The following induction hypothesis is used: There exists a constant K such that for $n = 1, \ldots, N$

$$\left\|\boldsymbol{u}_h^{(n)}\right\|_\infty, \left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty \le K. \tag{IH1}$$

The justification of (IH1) is established below.

Lemma 1 (Step 1a) *Assume (IH1) is true. For $\delta \le Ch$ and Δt sufficiently small there exists a unique solution $\boldsymbol{\sigma}_h^{(n+\theta)} \in S_h$ satisfying (4.2).*

Proof: Equation (4.2) can be equivalently written as

$$\mathcal{A}_1\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) = \frac{\lambda}{\theta \Delta t}\left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}\right) - \left((1-\omega)\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) - \lambda\left(\left(\boldsymbol{u}_h^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)} + g_a(\boldsymbol{\sigma}_h^n, \nabla \boldsymbol{u}_h^{(n)})\right), \boldsymbol{\tau}_{\delta^{(n)}}\right) + 2\alpha\left(\mathbf{d}(\boldsymbol{u}_h^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}}\right), \quad \forall\, \boldsymbol{\tau} \in S_h, \tag{4.8}$$

where

$$\mathcal{A}_1\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) := \frac{\lambda}{\theta \Delta t}\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) + \omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n)}}\right).$$

Here (4.8) represents a square linear system of equations $A\mathbf{x} = \mathbf{b}$. With the choice $\boldsymbol{\tau} = \boldsymbol{\sigma}_h^{(n+\theta)}$, examining the individual terms in $\mathcal{A}_1$ we see that

$$\begin{aligned}
\frac{\lambda}{\theta \Delta t}\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)}\right) &= \frac{\lambda}{\theta \Delta t}\left\|\boldsymbol{\sigma}_h^{(n+\theta)}\right\|^2, \\
\omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)}\right) &= \omega\left\|\boldsymbol{\sigma}_h^{(n+\theta)}\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\omega\delta\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{u}_h^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\theta)}\right) &= \frac{\omega\delta}{2}\left(\boldsymbol{u}_h^{(n)}, \nabla\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)}\right)\right) \\
&\le \frac{\omega\delta}{2}\left\|\boldsymbol{u}_h^{(n)}\right\|_\infty Ch^{-1}\left\|\boldsymbol{\sigma}_h^{(n+\theta)}\right\|^2 \\
&\le \frac{\omega\delta K C h^{-1}}{2}\left\|\boldsymbol{\sigma}_h^{(n+\theta)}\right\|^2.
\end{aligned}$$

Provided $\delta \le Ch$, and $\Delta t \le (2\lambda)/(\theta\omega KC)$ then $\mathcal{A}_1\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\sigma}_h^{(n+\theta)}\right) > 0$, and thus the $ker\,(\mathcal{A}_1) = \{\mathbf{0}\}$. It then follows that (4.2) has a unique solution.

$\blacksquare$

Lemma 2 (Step 1b) *There exists a unique solution* $\boldsymbol{u}_h^{(n+\theta)} \in Z_h$ *satisfying (4.3).*

Proof: Equation (4.3) can be equivalently written as

$$\mathcal{A}_2\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) = \frac{Re}{\theta\Delta t}\left(\boldsymbol{u}_h^{(n)}, \mathbf{v}\right) + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right), \ \ \forall\, \mathbf{v} \in Z_h,$$

where

$$\mathcal{A}_2\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) := \frac{Re}{\theta\Delta t}\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right).$$

Note that choosing $\mathbf{v} = \boldsymbol{u}_h^{(n+\theta)}$

$$\mathcal{A}_2\left(\boldsymbol{u}_h^{(n+\theta)}, \boldsymbol{u}_h^{(n+\theta)}\right) = \frac{Re}{\theta\Delta t}\left(\boldsymbol{u}_h^{(n+\theta)}, \boldsymbol{u}_h^{(n+\theta)}\right) + 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\boldsymbol{u}_h^{(n+\theta)})\right) > 0.$$

and existence and uniqueness of a solution to (4.3) has been shown.

$\blacksquare$

Lemma 3 (Step 2a) *There exists a unique solution* $\boldsymbol{u}_h^{(n+\tilde{\theta})} \in Z_h$ *satisfying (4.4).*

Proof: First write equation (4.4) as

$$\begin{aligned}\mathcal{A}_3\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \mathbf{v}\right) \ &= \ \frac{Re}{(1-2\theta)\Delta t}\left(\boldsymbol{u}_h^{(n+\theta)}, \mathbf{v}\right) - 2(1-\alpha)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right) \\ &\quad + \left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) - \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right),\end{aligned}$$

where

$$\mathcal{A}_3\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \mathbf{v}\right) := \frac{Re}{(1-2\theta)\Delta t}\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \mathbf{v}\right).$$

For the choice of $\mathbf{v} = \boldsymbol{u}^{(n+\tilde{\theta})}$, the system $\mathcal{A}_3\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}, \boldsymbol{u}_h^{(n+\tilde{\theta})}\right) > 0$, and the proof is complete.

$\blacksquare$

Lemma 4 (Step 2b) *Assume (IH1) is true. For* $\delta \le Ch$ *and* Δt *sufficiently small there exists a unique solution* $\boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \in S_h$ *satisfying (4.5).*

Proof: Write (4.5) as

$$\mathcal{A}_4\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}\right) = \frac{\lambda}{(1-2\theta)\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}\right) - \omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) + 2\alpha\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right),$$

with

$$\begin{aligned}\mathcal{A}_4\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}\right) &:= \frac{\lambda}{(1-2\theta)\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}\right) + (1-\omega)\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\ &\quad + \lambda\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) + \lambda\left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right).\end{aligned}$$

Estimating the terms in $\mathcal{A}_4\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right)$ yields

$$\frac{\lambda}{(1-2\theta)\Delta t}\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) = \frac{\lambda}{(1-2\theta)\Delta t}\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,$$

$$(1-\omega)\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) = (1-\omega)\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,$$

$$(1-\omega)\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \delta\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) \leq \epsilon_0\delta\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 + \frac{(1-\omega)^2\delta}{4\epsilon_0}\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,$$

$$\begin{aligned}
\lambda\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) &\leq \frac{\lambda\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty Ch^{-1}}{2}\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 \\
&\leq \frac{\lambda KCh^{-1}}{2}\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,
\end{aligned}$$

$$\lambda\left(\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \delta\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) = \lambda\delta\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,$$

$$\begin{aligned}
\lambda\left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}), \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) &\leq 4\lambda\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\| \\
&\leq 4\acute{d}^{1/2}\lambda\left\|\nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 \\
&\leq 4\acute{d}^{1/2}\lambda Ch^{-1}\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 \\
&\leq 4\acute{d}^{1/2}\lambda Ch^{-1}K\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2,
\end{aligned}$$

$$\begin{aligned}
\left(\lambda g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}), \delta\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right) &\leq 4\lambda\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|\left\|\delta\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\| \\
&\leq 4\acute{d}^{1/2}\lambda\left\|\nabla\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|\left\|\delta\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\| \\
&\leq \frac{4\acute{d}\lambda C^2h^{-2}K^2\delta}{\epsilon_1}\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 + \lambda\epsilon_1\delta\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{A}_4(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}) \geq &\left(\frac{\lambda}{(1-2\theta)\Delta t} + (1-\omega) - \frac{(1-\omega)^2\delta}{4\epsilon_0}\right. \\
&\left.-\lambda KCh^{-1}\left(\frac{1}{2} + 4\acute{d}^{1/2} + \frac{4\acute{d}KCh^{-1}\delta}{\epsilon_1}\right)\right)\left\|\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2 \\
&+\delta\left(\lambda - \epsilon_0 - \lambda\epsilon_1\right)\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}\right\|^2.
\end{aligned}$$

Choosing $\epsilon_0 = \frac{\lambda}{3}$, $\epsilon_1 = \frac{1}{3}$, $\delta \leq \mathcal{C}h^{-1}$, and $\Delta t \leq \mathcal{C}h^{-1}$ establishes $\mathcal{A}_4(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}) > 0$. Hence, a unique solution exists for (4.2).

■

The unique solvability of (4.6) and (4.7) of the algorithm follows exactly as (4.2) and (4.3).

4.3 Error Estimates

In this section supporting analysis for the θ-method described in Section 2 is given. The θ-method decouples the stress updates from the velocity-pressure updates. Though the modeling system is nonlinear, the updates for the stress and velocity-pressure only require linear systems of equations to be solved. Motivated by this decoupling (as a first step toward an analysis of the fully discretized system) we investigate separately the stress and velocity-pressure approximations. First a θ-method for the constitutive equation given by (4.2), (4.5), and (4.6) is investigated assuming the velocity and pressure are known. Then a θ-method for the Stokes-like problem given by (4.3), (4.4), and (4.7) is analyzed assuming the stress is known. A priori error estimates for each scheme are given in Theorems 4.1 and 4.2, and a discussion of the proofs is given below. (Detailed proofs are given in section 5, and 6.)

It is convenient to define the following notation:

$$
\begin{aligned}
\tilde{\boldsymbol{u}}_h &:= \text{ discrete approximation using true } \boldsymbol{\sigma}, \\
\tilde{\boldsymbol{\sigma}}_h &:= \text{ discrete approximation using true } \boldsymbol{u}, \text{ and } p.
\end{aligned}
$$

Theorem 4.1 (Assuming u and p are known) *For sufficiently smooth solutions $\boldsymbol{\sigma}$, $\boldsymbol{u}$, p such that*

$$
\|\boldsymbol{\sigma}\|_\infty, \|\boldsymbol{\sigma}_t\|_\infty, \|\boldsymbol{u}\|_\infty, \|\boldsymbol{u}_t\|_\infty, \|\boldsymbol{u}_{tt}\|_\infty, \|\nabla\boldsymbol{u}\|_\infty, \|(\nabla\boldsymbol{u})_t\|_\infty, \text{ and } \|(\nabla\boldsymbol{u})_{tt}\|_\infty \leq M, \quad \forall\, t \in [0,T],
$$

$\Delta t \leq Ch^2$, the fractional step θ-method approximation, $\tilde{\boldsymbol{\sigma}}_h$ given by Step 1a, Step 2b, and Step 3a converges to $\boldsymbol{\sigma}$ on the interval $(0,T]$ as $\Delta t, h \to 0$, and satisfies the error estimates:

$$
|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{\infty,0} \leq F_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \tag{4.9}
$$

$$
|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} \leq F_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \tag{4.10}
$$

and

$$
|||\boldsymbol{u}\cdot\nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)|||_{\tilde{\theta}} := \left(\sum_{n=1}^{N} \Delta t \left\| \boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}\right)\right\|^2\right)^{\frac{1}{2}} \leq F_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \tag{4.11}
$$

where

$$
\begin{aligned}
F_{\boldsymbol{\sigma}}(\Delta t, h, \delta) \;:=\; & C(\Delta t)^2\Big(\|\boldsymbol{\sigma}_{ttt}\|_{0,0} + \|\boldsymbol{\sigma}_{tt}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,1} \\
& \qquad + \|\boldsymbol{\sigma}_{tt}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + \|\boldsymbol{\sigma}\|_{0,0} + C_T\Big) \\
& + C(\Delta t)\delta\left(\|\boldsymbol{\sigma}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + C_T\right) \\
& + C\left(h^{m+1} + h^m + \delta h^m\right)|||\boldsymbol{\sigma}|||_{0,m+1} \\
& + Ch^{m+1}\|\boldsymbol{\sigma}_t\|_{0,m+1} + C\delta\,|||\boldsymbol{\sigma}_t|||_{0,0} + Ch^{m+1}\,|||\boldsymbol{\sigma}|||_{\infty,0}\,.
\end{aligned} \tag{4.12}
$$

Theorem 4.2 (Assuming $\boldsymbol{\sigma}$ is known) *For a sufficiently smooth solutions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, p such that $\|\boldsymbol{\sigma}\|_\infty \leq M, \ \ \forall t \in [0,T]$, and $\Delta t \leq Ch^2$, the fractional step θ-method approximation, $\tilde{\boldsymbol{u}}_h$ given by Step 1b, Step 2a, and Step 3b converges to $\boldsymbol{u}$ on the interval $(0,T]$ as $\Delta t, h \to 0$, and satisfies the error estimates:*

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{\infty,0} \leq F_{\boldsymbol{u}}(\Delta t, h, \delta), \tag{4.13}$$

and

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} \leq F_{\boldsymbol{u}}(\Delta t, h, \delta), \tag{4.14}$$

where

$$\begin{aligned} F_{\boldsymbol{u}}(\Delta t, h, \delta) \ \ := \ \ & Ch^{k+1}\,\|\boldsymbol{u}_t\|_{0,k+1} + Ch^k\,|||\boldsymbol{u}|||_{0,k+1} + Ch^{q+1}\,|||p|||_{0,q+1} \\ & + C(\Delta t)^2\,\|\boldsymbol{u}_{ttt}\|_{0,0} + C(\Delta t)^2\,\|\boldsymbol{u}_{tt}\|_{0,1} + C(\Delta t)^2\,\|\mathbf{f}_{tt}\|_{0,0} \\ & + C(\Delta t)^2 C_T + Ch^k\,|||\boldsymbol{u}|||_{\infty,k+1}\,. \end{aligned} \tag{4.15}$$

Outline of the proof. In [6], and [7] the analysis for a fractional step θ-method for the convection diffusion equation was presented. The proof of Theorems 4.1 and 4.2 are done in an analogous manner to the proof presented in [7]. Here we present an outline of the proof for Theorem 4.1, and note that the proof of Theorem 4.2 is accomplished in a similar manner.

Step 1. When obtaining a priori error estimates for the time dependent approximation schemes it is useful to examine a *unit stride*, i.e. the terms analyzed are distance Δt apart. For the θ-method this is accomplished by considering linear combinations of the approximation methods steps. In order to obtain *unit strides* from $\tilde{\boldsymbol{\sigma}}_h^{(n)}$ to $\tilde{\boldsymbol{\sigma}}_h^{(n+1)}$, $\tilde{\boldsymbol{\sigma}}_h^{(n-\theta)}$ to $\tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}$, and $\tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)}$ to $\tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}$ the following linear combinations are formed:

$$\theta\Delta t(4.2) + (1-2\theta)\,\Delta t(4.5) + \theta\Delta t(4.6), \tag{4.16}$$

$$\theta\Delta t(4.2) + (1-2\theta)\,\Delta t(4.5) + \theta\Delta t\left((4.6) \text{ with } n \to n-1\right), \tag{4.17}$$

$$\theta\Delta t(4.2) + (1-2\theta)\,\Delta t\left((4.5) \text{ with } n \to n-1\right) + \theta\Delta t\left((4.6) \text{ with } n \to n-1\right). \tag{4.18}$$

Step 2. Evaluate (3.7) at the midpoint of each *unit stride* and subtract equations (4.16)-(4.18). Rearrange terms to obtain the equations:

$$\begin{aligned} &\left(\left(\boldsymbol{\sigma}^{(n+1)} - \tilde{\boldsymbol{\sigma}}_h^{(n+1)}\right) - \left(\boldsymbol{\sigma}^{(n)} - \tilde{\boldsymbol{\sigma}}_h^{(n)}\right), \boldsymbol{\tau}\right) + \Delta t\mathcal{B}_{1_{pos}}\left(\left(\boldsymbol{\sigma}^{(n+1)} - \tilde{\boldsymbol{\sigma}}_h^{(n+1)}\right), \boldsymbol{\tau}\right) \\ &\qquad = \Delta t\mathcal{B}_{1_{rem}}\left(\Delta t, \boldsymbol{\sigma}, \boldsymbol{u}, \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}, \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}, \tilde{\boldsymbol{\sigma}}_h^{(n)}, \boldsymbol{\tau}\right), \end{aligned} \tag{4.19}$$

$$\begin{aligned} &\left(\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}\right) - \left(\boldsymbol{\sigma}^{(n-\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)}\right), \boldsymbol{\tau}\right) + \Delta t\mathcal{B}_{2_{pos}}\left(\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}\right), \boldsymbol{\tau}\right) \\ &\qquad = \Delta t\mathcal{B}_{2_{rem}}\left(\Delta t, \boldsymbol{\sigma}, \boldsymbol{u}, \tilde{\boldsymbol{\sigma}}_h^{(n+\tilde{\theta})}, \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}, \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)}, \tilde{\boldsymbol{\sigma}}_h^{(n)}, \boldsymbol{\tau}\right), \end{aligned} \tag{4.20}$$

$$\left(\left(\boldsymbol{\sigma}^{(n+\theta)}-\tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}\right)-\left(\boldsymbol{\sigma}^{(n+\theta-1)}-\tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)}\right),\boldsymbol{\tau}\right)+\Delta t\mathcal{B}_{3_{pos}}\left(\left(\boldsymbol{\sigma}^{(n+\theta)}-\tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}\right),\boldsymbol{\tau}\right)$$
$$=\Delta t\mathcal{B}_{3_{rem}}\left(\Delta t,\boldsymbol{\sigma},\boldsymbol{u},\tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)},\tilde{\boldsymbol{\sigma}}_h^{(n+\theta)},\tilde{\boldsymbol{\sigma}}_h^{(n-\theta)},\tilde{\boldsymbol{\sigma}}_h^{(n)},\boldsymbol{\tau}\right),\quad (4.21)$$

where $\mathcal{B}_{1_{pos}}$, $\mathcal{B}_{2_{pos}}$, and $\mathcal{B}_{3_{pos}}$ denote the positive part of the operators.

Step 3. Use $\boldsymbol{\sigma}^{(n)}-\tilde{\boldsymbol{\sigma}}_h^{(n)}=\tilde{\boldsymbol{\Gamma}}^{(n)}+\tilde{\boldsymbol{F}}^{(n)}$, and choose $\boldsymbol{\tau}=\tilde{\boldsymbol{F}}^{(n)}$, in (4.19) to obtain an expression of the form

$$\left\|\boldsymbol{F}^{(n+1)}\right\|^2-\left\|\boldsymbol{F}^{(n)}\right\|^2+\Delta t\mathcal{B}_{1_{pos}}\left(\tilde{\boldsymbol{F}}^{(n+1)},\tilde{\boldsymbol{F}}^{(n+1)}\right)\leq\Delta t\tilde{\mathcal{B}}_{1_{rem}}\left(\Delta t,\boldsymbol{\sigma},\boldsymbol{u},\tilde{\boldsymbol{\Gamma}},\tilde{\boldsymbol{F}}\right).\quad (4.22)$$

Assuming that $\boldsymbol{F}^{(0)}=0$, (4.22) is then summed from $n=0$ to $n=l-1$ so that the expression *telescopes* to

$$\left\|\boldsymbol{F}^{l}\right\|^2+\Delta t\sum_{n=0}^{l-1}\mathcal{B}_{1_{pos}}\left(\tilde{\boldsymbol{F}}^{(n+1)},\tilde{\boldsymbol{F}}^{(n+1)}\right)=\Delta t\mathcal{R}_1\left(\Delta t,\boldsymbol{\sigma},\boldsymbol{u},\tilde{\boldsymbol{\Gamma}},\tilde{\boldsymbol{F}}\right).\quad (4.23)$$

A similar approach is taken with (4.20) and (4.21) where $\boldsymbol{\tau}$ is chosen to be $\boldsymbol{F}^{(n+\tilde{\theta})}$ and $\boldsymbol{F}^{(n+\theta)}$, respectively, giving equations for $\left\|\boldsymbol{F}^{(l-\theta)}\right\|^2$, and $\left\|\boldsymbol{F}^{(l-1+\theta)}\right\|^2$. These three equations are then added together to form a single equation.

Step 4. Suitable inequalities/estimates are then applied to the terms in the equation.

Step 5. Gronwall's lemma is applied to get an estimate for $\left\|\tilde{\boldsymbol{F}}^{l-1+\theta}\right\|^2+\left\|\tilde{\boldsymbol{F}}^{l-\theta}\right\|^2+\left\|\tilde{\boldsymbol{F}}^{l}\right\|^2$, and then using the triangle inequality we obtain the error estimate for $\left\|\boldsymbol{\sigma}^{(l)}-\tilde{\boldsymbol{\sigma}}_h^{(l)}\right\|+\left\|\boldsymbol{\sigma}^{(l-\theta)}-\tilde{\boldsymbol{\sigma}}_h^{(l-\theta)}\right\|+\left\|\boldsymbol{\sigma}^{(l-1+\theta)}-\tilde{\boldsymbol{\sigma}}_h^{(l-1+\theta)}\right\|$.

■

In Section 7 the numerical results presented use continuous, piecewise linear approximations for $\boldsymbol{\sigma}$, and p, and a continuous, piecewise quadratic approximation for $\boldsymbol{u}$. With these approximations we have the following estimates:

Corollary 1 *For S_h the space of continuous, piecewise linear functions, $\Delta t\leq Ch^2$, and $\boldsymbol{\sigma},\boldsymbol{u},p$ sufficiently smooth, the approximation $\tilde{\boldsymbol{\sigma}}_h$ satisfies the error estimate:*

$$|||\boldsymbol{u}\cdot\nabla\left(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h\right)|||_{\tilde{\theta}}\leq C\left((\Delta t)^2+\Delta t\delta+h\delta+h+\delta\right).\quad (4.24)$$

Corollary 2 *For X_h the space of continuous, piecewise quadratic functions, and Q_h the space of continuous, piecewise linear functions, $\Delta t\leq Ch^2$, and $\boldsymbol{\sigma},\boldsymbol{u},p$ sufficiently smooth, the approximation $\tilde{\boldsymbol{u}}_h$ satisfies the error estimate:*

$$|||\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|||_{0,1}\leq C\left((\Delta t)^2+h^2\right).\quad (4.25)$$

5 Proof of Theorem 4.1

In this Section the proof of Theorem 4.1 is accomplished using the steps outlined in Section 4. We define the upwinded test element using true solution $\boldsymbol{u}^{(n)}$ as :

$$\boldsymbol{\tau}_{\delta^{(n)}} := \boldsymbol{\tau} + \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\tau}. \tag{5.1}$$

For notational simplicity the 'tilde' is dropped from $\tilde{\boldsymbol{\sigma}}_h$, the discrete approximation using the true $\boldsymbol{u}$, and p.

5.1 Set up linear combinations:

Three linear combinations are set up as stated in (4.16), (4.17), and (4.18).

Linear combination 1: The first linear combination is given by

$$\theta(4.2) + (1-2\theta)\,(4.5) + \theta(4.6), \tag{5.2}$$

and yields the espression:

$$\begin{aligned}
\mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) &:= \lambda\left(d_t\left(\boldsymbol{\sigma}_h^{(n+1)}\right), \boldsymbol{\tau}\right) + \theta\omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) \\
&+ (1-2\theta)\,\omega\left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) + \theta\omega\left(\boldsymbol{\sigma}_h^{(n+1)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\
&+ \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) + \theta\tilde{\omega}\left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) + \ \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\
&+\theta\lambda\left(\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}}\right) + \ \tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\
&+\theta\lambda\left(g_a(\boldsymbol{\sigma}_h^n, \nabla\boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}}\right) - \ (1-\theta)2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}\right) \\
&-\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}}\right) = 0, \qquad \forall\, \boldsymbol{\tau} \in S_h.
\end{aligned} \tag{5.3}$$

The true solution for the constitutive equation evaluated at $t_{n+\frac{1}{2}}$ satisfies:

$$\begin{aligned}
0 = -\lambda\left(\frac{\partial\boldsymbol{\sigma}^{(n+\frac{1}{2})}}{\partial t}, \boldsymbol{\tau}\right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}\right) + 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}\right) \\
- \lambda\left(\boldsymbol{u}^{(n+\frac{1}{2})} \cdot \nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})} + g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla\boldsymbol{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}\right), \ \ \forall\, \boldsymbol{\tau} \in S.
\end{aligned} \tag{5.4}$$

An appropriatly upwinded version of the true soulution is obtained by considering

$$\theta * (5.4)\Big|_{\boldsymbol{\tau}_{\delta^{(n)}}} + \tilde{\theta} * (5.4)\Big|_{\boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}}}. \tag{5.5}$$

Next we add and subtract terms to expression (5.5) yielding:

$$\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \mathcal{I}_1^n(\boldsymbol{\tau}), \ \ \forall\, \boldsymbol{\tau} \in S, \tag{5.6}$$

where

$$
\begin{aligned}
\mathcal{I}_1^n(\boldsymbol{\tau}) := {}& \lambda \left(d_t \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \boldsymbol{\tau} \right) + \lambda\theta \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\tau} \right) \\
& + \lambda\tilde{\theta} \left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta \boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\tau} \right) + \omega\theta \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n)}} \right) \\
& + \omega\theta \left(\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) + \omega\left(1-2\theta\right) \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \tilde{\omega}\theta \left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n)}} \right) + \tilde{\omega}\tilde{\theta} \left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \theta\lambda \left(\boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n)}} \right) \\
& + \tilde{\theta}\lambda \left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \theta\lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}_{\delta^{(n)}} \right) \\
& + \tilde{\theta}\lambda \left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\frac{1}{2})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \theta 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}} \right) + \tilde{\theta} 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right). \qquad (5.7)
\end{aligned}
$$

Subtraction of (5.3) from (5.6) gives:

$$
\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \mathcal{I}_1^n(\boldsymbol{\tau}), \ \forall\, \boldsymbol{\tau} \in S_h, \qquad (5.8)
$$

the first expression that will be bounded to complete the analysis.

Linear combination 2 : The second linear combination gives a full discrete time step of size Δt between $t_{n-\theta}$ and $t_{n+1-\theta}$ once we multiply the resulting expression by Δt in a later step. Using (4.17) divided by Δt yields,

$$
\begin{aligned}
\mathcal{B}_2^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) := {}& \lambda \left(d_t \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})} \right), \boldsymbol{\tau} \right) + \theta\omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) + (1-2\theta)\,\omega \left(\boldsymbol{\sigma}_h^{(n+\theta)}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \theta\omega \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n-\theta)}} \right) + (1-2\theta)\,\tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) + \theta\tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) \\
& + \theta\tilde{\omega} \left(\boldsymbol{\sigma}_h^{(n-\theta)}, \boldsymbol{\tau}_{\delta^{(n-\theta)}} \right) + (1-2\theta)\,\lambda \left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& + \theta\lambda \left(\boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}_h^{(n)}, \boldsymbol{\tau}_{\delta^{(n)}} \right) + \theta\lambda \left(\boldsymbol{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}_h^{(n-\theta)}, \boldsymbol{\tau}_{\delta^{(n-\theta)}} \right) \\
& + (1-2\theta)\,\lambda \left(g_a(\boldsymbol{\sigma}_h^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) + \theta\lambda \left(g_a(\boldsymbol{\sigma}_h^{(n)}, \nabla \boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}} \right) \\
& + \theta\lambda \left(g_a(\boldsymbol{\sigma}_h^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta^{(n-\theta)}} \right) - (1-2\theta)\, 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{\tau}_{\delta^{(n+\tilde{\theta})}} \right) \\
& - \theta 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta^{(n)}} \right) - \theta 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta^{(n-\theta)}} \right) = 0, \qquad \forall\, \boldsymbol{\tau} \in S_h. \qquad (5.9)
\end{aligned}
$$

The true solution to the constitutive equation evaluated at the midpoint between $t_{n-\theta}$ and $t_{n+1-\theta}$ is

$$
\begin{aligned}
0 = {}& -\lambda \left(\frac{\partial \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}}{\partial t}, \boldsymbol{\tau} \right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \boldsymbol{\tau} \right) + 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}), \boldsymbol{\tau} \right) \\
& - \lambda \left(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)} + g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}), \boldsymbol{\tau} \right), \ \ \forall\, \boldsymbol{\tau} \in S. \qquad (5.10)
\end{aligned}
$$

In a manner similar to (5.5) we form the linear combination

$$\theta(5.10)\Big|_{\boldsymbol{\tau}_{\delta(n)}} + (1-2\theta)\,(5.10)\Big|_{\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}} + \theta(5.10)\Big|_{\boldsymbol{\tau}_{\delta(n-\theta)}}, \tag{5.11}$$

and write the expression as

$$\mathcal{B}_2^n(\boldsymbol{\sigma},\boldsymbol{\tau}) = \mathcal{I}_2^n(\boldsymbol{\tau}),\ \forall\, \boldsymbol{\tau}\in S, \tag{5.12}$$

where

$$\begin{aligned}
\mathcal{I}_2^n(\boldsymbol{\tau}) :=\ & \lambda\left(d_t\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}\right) + \lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\tau}\right)\\
&+\lambda\,(1-2\theta)\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)},\delta\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\tau}\right) + \lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\tau}\right)\\
&+\theta\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n)}\right) + (1-2\theta)\,\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}\right)\\
&+\theta\omega\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n-\theta)}\right) + (1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}\right)\\
&+\theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n)}\right) + \theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n-\theta)}\right)\\
&+(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}\right)\\
&+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n)}\right)\\
&+\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\boldsymbol{\tau}_{\delta(n-\theta)}\right)\\
&+(1-2\theta)\,\lambda\left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}),\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}\right)\\
&+\theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}),\boldsymbol{\tau}_{\delta(n)}\right)\\
&+\theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)},\nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}),\boldsymbol{\tau}_{\delta(n-\theta)}\right)\\
&+\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n)}),\boldsymbol{\tau}_{\delta(n)}\right) + (1-2\theta)\,2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{\tau}_{\delta(n+\tilde{\theta})}\right)\\
&+\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\boldsymbol{\tau}_{\delta(n-\theta)}\right).
\end{aligned} \tag{5.13}$$

An error expression is now formed by subtracting (5.9) from (5.12):

$$\mathcal{B}_2^n(\boldsymbol{\sigma},\boldsymbol{\tau}) - \mathcal{B}_2^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) = \mathcal{I}_2^n(\boldsymbol{\tau}),\ \forall\,\boldsymbol{\tau}\in S_h. \tag{5.14}$$

Linear combination 3: We form the third linear combination (such that when multiplied by Δt a full discrete time step between $t_{n+\theta-1}$ and $t_{n+\theta}$ may be analyzed) by considering

$$\theta * (4.2) + (1-2\theta) * ((4.5) \text{ with } n\to n-1) + \theta * ((4.6) \text{ with } n\to n-1)\,. \tag{5.15}$$

Using (5.15) we obtain

$$\begin{aligned}
\mathcal{B}_3^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) :=\ & \lambda\left(d_t\left(\boldsymbol{\sigma}_h^{(n+\theta)}\right),\boldsymbol{\tau}\right) + \theta\omega\left(\boldsymbol{\sigma}_h^{(n+\theta)},\boldsymbol{\tau}_{\delta(n)}\right) + \theta\omega\left(\boldsymbol{\sigma}_h^{(n)},\boldsymbol{\tau}_{\delta(n-\theta)}\right)\\
&+(1-2\theta)\,\omega\left(\boldsymbol{\sigma}_h^{(n+\theta-1)},\boldsymbol{\tau}_{\delta(n-\theta)}\right) + \theta\tilde{\omega}\left(\boldsymbol{\sigma}_h^{(n)},\boldsymbol{\tau}_{\delta(n)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\sigma}_h^{(n-\theta)},\boldsymbol{\tau}_{\delta(n-\theta)}\right)\\
&+\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}_h^{(n-\theta)},\boldsymbol{\tau}_{\delta(n-\theta)}\right) + \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}_h^{(n)},\boldsymbol{\tau}_{\delta(n)}\right)\\
&+\tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}_h^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{\tau}_{\delta(n-\theta)}\right) + \theta\lambda\left(g_a(\boldsymbol{\sigma}_h^{n},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{\tau}_{\delta(n)}\right)\\
&-\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n)}),\boldsymbol{\tau}_{\delta(n)}\right) - \tilde{\theta}2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\boldsymbol{\tau}_{\delta(n-\theta)}\right) = 0, \qquad \forall\,\boldsymbol{\tau}\in S_h.
\end{aligned} \tag{5.16}$$

Evaluating the true solution at the midpoint between $t_{n+\theta-1}$ and $t_{n+\theta}$:

$$0 = -\lambda\left(\frac{\partial \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}}{\partial t}, \boldsymbol{\tau}\right) - \left(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}\right) + 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \boldsymbol{\tau}\right)$$
$$- \lambda\left(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)} \cdot \nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)} + g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \boldsymbol{\tau}\right), \quad \forall\, \boldsymbol{\tau} \in S. \quad (5.17)$$

An appropriately upwinded true solution is found using the linear combination

$$\theta(5.17)\Big|_{\boldsymbol{\tau}_{\delta(n)}} + \tilde{\theta}(5.17)\Big|_{\boldsymbol{\tau}_{\delta(n-\theta)}}. \quad (5.18)$$

Adding $\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau})$ to both sides of expression (5.18) gives

$$\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \mathcal{I}_3^n(\boldsymbol{\tau}),\ \forall\, \boldsymbol{\tau} \in S, \quad (5.19)$$

where

$$\begin{aligned}
\mathcal{I}_3^n(\boldsymbol{\tau}) := {} & \lambda\left(d_t\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}\right) + \lambda\theta\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{\tau}\right) \\
& + \lambda\tilde{\theta}\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{\tau}\right) + \theta\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n)}\right) \\
& + \theta\omega\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n-\theta)}\right) + (1-2\theta)\,\omega\left(\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n-\theta)}\right) \\
& + \theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n-\theta)}\right) \\
& + \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)} \cdot \nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n-\theta)}\right) \\
& + \theta\lambda\left(\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)} \cdot \nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{\tau}_{\delta(n)}\right) \\
& + \tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla\boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \boldsymbol{\tau}_{\delta(n-\theta)}\right) \\
& + \theta\lambda\left(g_a(\boldsymbol{\sigma}^{n}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \boldsymbol{\tau}_{\delta(n)}\right) \\
& + \theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \boldsymbol{\tau}_{\delta(n)}\right) \\
& + \tilde{\theta} 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \boldsymbol{\tau}_{\delta(n-\theta)}\right).
\end{aligned} \quad (5.20)$$

Subtraction of (5.16) from (5.19) yields the final error expression needed for analysis of the constitutive equation

$$\mathcal{B}_3^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_3^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \mathcal{I}_3^n(\boldsymbol{\tau}),\ \forall\, \boldsymbol{\tau} \in S_h. \quad (5.21)$$

Bounds are now found for (5.8),(5.14), and (5.21).

5.1.1 Bounding $\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$

In this section bounds on the term $\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{F}^{(n+1)}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{F}^{(n+1)})$ in expression (5.8) are examined. Recall for this proof we have dropped the 'tilde' from $\tilde{\boldsymbol{\sigma}}_h$. Thus,

$$\boldsymbol{\Gamma}^{(n)} = \boldsymbol{\sigma}^{(n)} - \mathcal{S}^{(n)}, \qquad \text{and} \qquad \boldsymbol{F}^{(n)} = \mathcal{S}^{(n)} - \tilde{\boldsymbol{\sigma}}_h^{(n)}.$$

It is also convenient to use the notation:

$$\boldsymbol{F}^{(\nu)}_{\delta^{(\mu)}} := \boldsymbol{F}^{(\nu)} + \delta \boldsymbol{u}^{(\mu)} \cdot \nabla \boldsymbol{F}^{(\nu)}. \tag{5.22}$$

Here,

$$\begin{aligned}
\mathcal{B}_1^n(\boldsymbol{\sigma}, \boldsymbol{F}^{(n+1)}) - \mathcal{B}_1^n(\boldsymbol{\sigma}_h, \boldsymbol{F}^{(n+1)}) &= \lambda \left(d_t(e_{\boldsymbol{\sigma}}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right) + \ \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n)}} \right) \\
&+ \theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+1)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n+\tilde{\theta})}} \right) + \ (1-2\theta)\,\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n+\tilde{\theta})}} \right) \\
&+ \tilde{\theta}\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}_{\delta^{(n+\tilde{\theta})}} \right) + \ \theta\tilde{\omega} \left(e_{\boldsymbol{\sigma}}^{(n)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n)}} \right) \\
&+ \tilde{\theta}\lambda \left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}_{\delta^{(n+\tilde{\theta})}} \right) + \ \theta\lambda \left(\boldsymbol{u}^{(n)} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n)}} \right) \\
&+ \tilde{\theta}\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}), \boldsymbol{F}^{(n+1)}_{\delta^{(n+\tilde{\theta})}} \right) + \ \theta\lambda \left(g_a(e_{\boldsymbol{\sigma}}^{(n)}, \nabla \boldsymbol{u}^{(n)}), \boldsymbol{F}^{(n+1)}_{\delta^{(n)}} \right).
\end{aligned} \tag{5.23}$$

Each of the terms in (5.23) is now bounded.

$$\lambda \left(d_t(e_{\boldsymbol{\sigma}}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right) \ = \ \lambda \left(d_t(\boldsymbol{F}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right) + \ \lambda \left(d_t(\boldsymbol{\Gamma}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right), \tag{5.24}$$

where

$$\begin{aligned}
\lambda \left(d_t(\boldsymbol{F}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right) &= \frac{\lambda}{\Delta t} \left(\left(\boldsymbol{F}^{(n+1)}, \boldsymbol{F}^{(n+1)} \right) - \left(\boldsymbol{F}^{(n)}, \boldsymbol{F}^{(n+1)} \right) \right) \\
&\geq \frac{\lambda}{\Delta t} \left(\left\| \boldsymbol{F}^{(n+1)} \right\|^2 - \left\| \boldsymbol{F}^{(n+1)} \right\| \left\| \boldsymbol{F}^{(n)} \right\| \right) \\
&\geq \frac{\lambda}{2\Delta t} \left(\left\| \boldsymbol{F}^{(n+1)} \right\|^2 - \left\| \boldsymbol{F}^{(n)} \right\|^2 \right),
\end{aligned} \tag{5.25}$$

and,

$$\lambda \left(d_t(\boldsymbol{\Gamma}^{(n+1)}), \boldsymbol{F}^{(n+1)} \right) \leq \frac{\lambda}{4} \left\| d_t \left(\boldsymbol{\Gamma}^{(n+1)} \right) \right\|^2 + \lambda \left\| \boldsymbol{F}^{(n+1)} \right\|^2. \tag{5.26}$$

Each of the next eight terms in (5.23) is rewritten as the sum of four terms. Starting with

$$\begin{aligned}
\theta\omega \left(e_{\boldsymbol{\sigma}}^{(n+\theta)}, \boldsymbol{F}^{(n+1)}_{\delta^{(n)}} \right) &= \theta\omega \left(\boldsymbol{F}^{(n+\theta)}, \boldsymbol{F}^{(n+1)} \right) + \theta\omega \left(\boldsymbol{F}^{(n+\theta)}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right) \\
&+ \theta\omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \boldsymbol{F}^{(n+1)} \right) + \theta\omega \left(\boldsymbol{\Gamma}^{(n+\theta)}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right).
\end{aligned} \tag{5.27}$$

we have the bounds

$$\begin{aligned}
\theta\omega \left(\boldsymbol{F}^{(n+\theta)}, \boldsymbol{F}^{(n+1)} \right) &\leq \theta\omega \left\| \boldsymbol{F}^{(n+\theta)} \right\| \left\| \boldsymbol{F}^{(n+1)} \right\| \\
&\leq \frac{\theta\omega}{2} \left\| \boldsymbol{F}^{(n+\theta)} \right\|^2 + \frac{\theta\omega}{2} \left\| \boldsymbol{F}^{(n+1)} \right\|^2,
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
\theta\omega \left(\boldsymbol{F}^{(n+\theta)}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right) &\leq \theta\omega \left\| \boldsymbol{F}^{(n+\theta)} \right\| \left\| \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right\| \\
&\leq \delta\theta\omega \left\| \boldsymbol{F}^{(n+\theta)} \right\| \left\| \boldsymbol{u}^{(n)} \right\|_\infty \acute{d}^{\frac{1}{2}} \left\| \nabla \boldsymbol{F}^{(n+1)} \right\| \\
&\leq \delta\theta\omega M \acute{d}^{\frac{1}{2}} \left\| \boldsymbol{F}^{(n+\theta)} \right\| \left\| \nabla \boldsymbol{F}^{(n+1)} \right\| \\
&\leq \frac{\theta\omega M^2 \acute{d}}{4} \left\| \boldsymbol{F}^{(n+\theta)} \right\|^2 + \delta^2\theta\omega \left\| \nabla \boldsymbol{F}^{(n+1)} \right\|^2,
\end{aligned} \tag{5.29}$$

$$\theta\omega\left(\mathbf{\Gamma}^{(n+\theta)},\boldsymbol{F}^{(n+1)}\right) \;\le\; \frac{\theta\omega}{2}\left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.30}$$

and

$$\theta\omega\left(\mathbf{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \;\le\; \frac{\theta\omega M^2\acute{d}}{4}\left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.31}$$

Splitting the next term into its pieces gives:

$$\begin{aligned}\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n+1)},\boldsymbol{F}_{\delta^{(n+\tilde\theta)}}^{(n+1)}\right) \;&=\; \theta\omega\left(\boldsymbol{F}^{(n+1)},\boldsymbol{F}^{(n+1)}\right)+\theta\omega\left(\boldsymbol{F}^{(n+1)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\\ &\quad+\theta\omega\left(\mathbf{\Gamma}^{(n+1)},\boldsymbol{F}^{(n+1)}\right)+\theta\omega\left(\mathbf{\Gamma}^{(n+1)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right).\end{aligned} \tag{5.32}$$

Bounding each term we have

$$\theta\omega\left(\boldsymbol{F}^{(n+1)},\boldsymbol{F}^{(n+1)}\right)=\theta\omega\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.33}$$

$$\theta\omega\left(\boldsymbol{F}^{(n+1)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\le\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+1)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.34}$$

$$\theta\omega\left(\mathbf{\Gamma}^{(n+1)},\boldsymbol{F}^{(n+1)}\right)\le\frac{\theta\omega}{2}\left\|\mathbf{\Gamma}^{(n+1)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.35}$$

and

$$\theta\omega\left(\mathbf{\Gamma}^{(n+1)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\le\frac{\theta\omega M^2\acute{d}}{4}\left\|\mathbf{\Gamma}^{(n+1)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.36}$$

Next

$$\begin{aligned}(1-2\theta)\,\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n+\tilde\theta)}}^{(n+1)}\right) \;&=\; (1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{(n+1)}\right)\\ &\quad+(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\\ &\quad+(1-2\theta)\,\omega\left(\mathbf{\Gamma}^{(n+\theta)},\boldsymbol{F}^{(n+1)}\right)\\ &\quad+(1-2\theta)\,\omega\left(\mathbf{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right).\end{aligned} \tag{5.37}$$

Bounding each term

$$(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{(n+1)}\right)\le\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.38}$$

$$\begin{aligned}(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \;&\le\; \frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2\\ &\quad+\;(1-2\theta)\,\omega\delta^2\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2\end{aligned} \tag{5.39}$$

$$(1-2\theta)\,\omega\left(\mathbf{\Gamma}^{(n+\theta)},\boldsymbol{F}^{(n+1)}\right)\le\frac{(1-2\theta)\,\omega}{2}\left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2 \tag{5.40}$$

$$\begin{aligned}(1-2\theta)\,\omega\left(\mathbf{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n+\tilde\theta)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \;&\le\; \frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2\\ &\quad+\delta^2\,(1-2\theta)\,\omega\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2\end{aligned} \tag{5.41}$$

Splitting the following gives:

$$\tilde{\theta}\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+1)}\right) = \tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right),$$

and these terms are bounded as

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.42}$$

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.43}$$

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.44}$$

and

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \delta^2\tilde{\theta}\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.45}$$

Continuing

$$\begin{aligned}\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)}, \boldsymbol{F}_{\delta^{(n)}}^{(n+1)}\right) &= \theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \boldsymbol{F}^{(n+1)}\right) + \theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\\ &\quad+\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \boldsymbol{F}^{(n+1)}\right) + \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right).\end{aligned} \tag{5.46}$$

These terms are bounded as:

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.47}$$

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.48}$$

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.49}$$

and

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.50}$$

Next the convective terms are handled. Thus,

$$\begin{aligned}\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+1)}\right) &= \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right)\\ &+ \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right)\\ &+ \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right).\end{aligned} \tag{5.51}$$

We bound each part in the following manner:

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+1)}\right\|^2 + \tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \tag{5.52}$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\delta\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2 + \frac{\delta\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.53)$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\boldsymbol{F}^{(n+1)}\right) \leq \frac{\tilde{\theta}\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+1)}\right\|^2 + \tilde{\theta}\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2, \quad (5.54)$$

and

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\delta^2\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2 + \frac{\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \quad (5.55)$$

Similarly the second convective term is split as:

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+1)}\right) = \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+1)}\right) + \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) + \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+1)}\right) + \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right), \quad (5.56)$$

and bounded term by term with

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+1)}\right\|^2 + \theta\lambda\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2, \quad (5.57)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\delta\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2 + \frac{\delta\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2, \quad (5.58)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+1)}\right\|^2 + \theta\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2, \quad (5.59)$$

and

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \leq \frac{\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2 + \frac{\delta^2\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2. \quad (5.60)$$

Lastly the $g_a(\cdot,\cdot)$ terms are considered. Splitting,

$$\tilde{\theta}\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+1)}\right) = \tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right) + \tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right), \quad (5.61)$$

and bounding each of these terms gives:

$$\begin{aligned}\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right) &\leq \tilde{\theta}\lambda\left\|g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})\right\|\left\|\boldsymbol{F}^{(n+1)}\right\| \\ &\leq 4\acute{d}\tilde{\theta}\lambda\left\|\nabla\boldsymbol{u}^{(n+\tilde{\theta})}\right\|_\infty\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|\left\|\boldsymbol{F}^{(n+1)}\right\| \\ &\leq 4\acute{d}\tilde{\theta}\lambda M\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|\left\|\boldsymbol{F}^{(n+1)}\right\| \\ &\leq 4\tilde{\theta}\lambda\acute{d}M\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2 + \tilde{\theta}\lambda\acute{d}M\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.62)\end{aligned}$$

$$
\begin{aligned}
\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) &\leq \tilde{\theta}\lambda\delta\left\|g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})\right\|\left\|\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right\| \\
&\leq 4\acute{d}^2\tilde{\theta}\lambda\delta\left\|\nabla\boldsymbol{u}^{(n+\tilde{\theta})}\right\|_\infty^2\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|\left\|\nabla\boldsymbol{F}^{(n+1)}\right\| \\
&\leq 4\acute{d}^2\tilde{\theta}\lambda\delta M^2\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|\left\|\nabla\boldsymbol{F}^{(n+1)}\right\| \\
&\leq 4\acute{d}^4\tilde{\theta}\lambda M^4\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\delta^2\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.63)
\end{aligned}
$$

$$
\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right)\leq 4\tilde{\theta}\lambda\acute{d}M\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2+\tilde{\theta}\lambda\acute{d}M\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.64}
$$

and

$$
\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq 4\acute{d}^4\tilde{\theta}\lambda M^4\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2+\delta^2\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.65}
$$

Similarly,

$$
\begin{aligned}
\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}_{\delta^{(n)}}^{(n+1)}\right) &= \theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+1)}\right) \\
&\quad+\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+1)}\right) \\
&\quad+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right), \quad (5.66)
\end{aligned}
$$

and has the following bounds:

$$
\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+1)}\right)\leq 4\theta\lambda\acute{d}M\left\|\boldsymbol{F}^{(n)}\right\|^2+\theta\lambda\acute{d}M\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.67}
$$

$$
\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq 4\acute{d}^4\theta\lambda M^4\left\|\boldsymbol{F}^{(n)}\right\|^2+\theta\lambda\delta^2\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.68}
$$

$$
\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+1)}\right)\leq 4\theta\lambda\acute{d}M\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\theta\lambda\acute{d}M\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.69}
$$

and

$$
\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq 4\theta\lambda\acute{d}^4M^4\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2. \tag{5.70}
$$

5.1.2 Bounding $\mathcal{B}_2^n(\boldsymbol{\sigma},\boldsymbol{\tau})-\mathcal{B}_2^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau})$

We continue in this section by determining bounds on the terms $\mathcal{B}_2^n(\boldsymbol{\sigma},\boldsymbol{\tau})-\mathcal{B}_2^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau})$ in expression (5.14). Note that the value of $\boldsymbol{\tau}$ is chosen to be $\boldsymbol{F}^{(n+\tilde{\theta})}$. Thus,

$$
\begin{aligned}
\mathcal{B}_2^n(\boldsymbol{\sigma},\boldsymbol{F}^{(n+\tilde{\theta})})-\mathcal{B}_2^n(\boldsymbol{\sigma}_h,\boldsymbol{F}^{(n+\tilde{\theta})}) &= \lambda\left(d_te_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\boldsymbol{F}^{(n+\tilde{\theta})}\right)+\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right) \\
&\quad+(1-2\theta)\,\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n+\theta)}}^{(n+\tilde{\theta})}\right)+\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})}\right)+(1-2\theta)\,\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})}\right) \\
&\quad+\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right)+\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})}\right)+(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})}\right) \\
&\quad+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right)+\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})}\right) \\
&\quad+(1-2\theta)\,\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})}\right)+\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right) \\
&\quad+\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})}\right). \quad (5.71)
\end{aligned}
$$

Each term in (5.71) is now bounded. Here

$$\lambda\left(d_t\left(e_{\boldsymbol{\sigma}}^{\left(n+\tilde{\theta}\right)}\right),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)=\lambda\left(d_t\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\lambda\left(d_t\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right), \tag{5.72}$$

with

$$\lambda\left(d_t\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\geq\frac{\lambda}{2\Delta t}\left(\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2-\left\|\boldsymbol{F}^{\left(n-\theta\right)}\right\|^2\right), \tag{5.73}$$

and

$$\lambda\left(d_t\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\lambda\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\frac{\lambda}{4}\left\|d_t\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right)\right\|^2. \tag{5.74}$$

Then

$$\begin{aligned}\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n)}}^{\left(n+\tilde{\theta}\right)}\right)&=\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\end{aligned} \tag{5.75}$$

and the terms are bounded as:

$$\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.76}$$

$$\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.77}$$

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|, \tag{5.78}$$

and

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \tag{5.79}$$

Splitting the next term:

$$\begin{aligned}(1-2\theta)\,\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{\left(n+\tilde{\theta}\right)}}^{\left(n+\tilde{\theta}\right)}\right)&=(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right).\end{aligned} \tag{5.80}$$

Bounding these terms:

$$(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.81}$$

$$\begin{aligned}&(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\qquad\leq\frac{(1-2\theta)\,\omega}{4\epsilon_1}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\delta^2\,(1-2\theta)\,\omega\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2,\end{aligned} \tag{5.82}$$

$$(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.83}$$

and

$$(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\omega}{4\epsilon_1}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|^2+\delta^2\,(1-2\theta)\,\omega\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \quad (5.84)$$

Next

$$\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{\left(n+\tilde{\theta}\right)}\right)=\theta\omega\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\omega\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$+\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right), \quad (5.85)$$

and each term is bounded as

$$\theta\omega\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.86)$$

$$\theta\omega\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.87)$$

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.88)$$

and

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \quad (5.89)$$

The term

$$(1-2\theta)\,\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}_{\delta^{\left(n+\tilde{\theta}\right)}}^{\left(n+\tilde{\theta}\right)}\right)=(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$+(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$+(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right), \quad (5.90)$$

is bounded as

$$(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)=(1-2\theta)\,\tilde{\omega}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.91)$$

$$(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\tilde{\omega}}{4\epsilon_1}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\delta^2\,(1-2\theta)\,\tilde{\omega}\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.92)$$

$$(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq\frac{(1-2\theta)\,\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right\|^2+\frac{(1-2\theta)\,\tilde{\omega}}{2}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.93)$$

and

$$(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\tilde{\omega}}{4\epsilon_1}\left\|\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right\|^2+\delta^2\,(1-2\theta)\,\tilde{\omega}\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \quad (5.94)$$

The term

$$\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)}, \boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right) = \theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) + \theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \delta\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)$$
$$+ \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) + \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right), \quad (5.95)$$

with bounds

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.96)$$

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)}, \delta\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.97)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.98)$$

and

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)}, \delta\boldsymbol{u}^{(n)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2. \quad (5.99)$$

Considering,

$$\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n-\theta)}, \boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\tilde{\theta})}\right) = \theta\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) + \theta\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)}, \delta\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)$$
$$+ \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) + \theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)}, \delta\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right). \quad (5.100)$$

We have the bounds

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.101)$$

$$\theta\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)}, \delta\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.102)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2 + \frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \quad (5.103)$$

and

$$\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)}, \delta\boldsymbol{u}^{(n-\theta)} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \leq \frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2 + \delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2. \quad (5.104)$$

Examining the convective terms:

$$(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})}, \boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})}\right) = (1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right)$$
$$+ (1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) + (1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right)$$
$$+ (1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{\Gamma}^{(n+\tilde{\theta})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right). \quad (5.105)$$

Bounding each piece, using $\boldsymbol{u}\big|_{\partial\Omega} = 0$, and $\nabla \cdot \boldsymbol{u}^{(n+\tilde{\theta})} = 0$

$$\begin{aligned}
(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) &= -(1-2\theta)\,\lambda\left(\nabla \cdot \boldsymbol{u}^{(n+\tilde{\theta})}\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&\quad -(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&= -(1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla\boldsymbol{F}^{(n+\tilde{\theta})}, \boldsymbol{F}^{(n+\tilde{\theta})}\right).
\end{aligned}$$

Thus,

$$(1-2\theta)\,\lambda\left(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)=0, \tag{5.106}$$

$$(1-2\theta)\,\lambda\left(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)=\delta\,(1-2\theta)\,\lambda\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.107}$$

$$\begin{aligned}&(1-2\theta)\,\lambda\left(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\qquad\le\frac{(1-2\theta)\,\lambda M^2\acute{d}}{4}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+(1-2\theta)\,\lambda\left\|\nabla\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right\|^2,\end{aligned} \tag{5.108}$$

and

$$\begin{aligned}&(1-2\theta)\,\lambda\left(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\qquad\le\delta^2\,(1-2\theta)\,\lambda\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\frac{(1-2\theta)\,\lambda M^2\acute{d}}{4\epsilon_1}\left\|\nabla\boldsymbol{\Gamma}^{\left(n+\tilde{\theta}\right)}\right\|^2.\end{aligned} \tag{5.109}$$

The next convective term is split as

$$\begin{aligned}\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{\left(n+\tilde{\theta}\right)}\right)&=\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\end{aligned} \tag{5.110}$$

and bounded by

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\le\frac{\theta\lambda M^2\acute{d}}{4}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2, \tag{5.111}$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\le\frac{\delta\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\frac{\delta\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2, \tag{5.112}$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\le\frac{\theta\lambda M^2\acute{d}}{4}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2, \tag{5.113}$$

and

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\le\frac{\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\frac{\delta^2\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2. \tag{5.114}$$

Next

$$\begin{aligned}\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{\left(n+\tilde{\theta}\right)}\right)&=\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&\quad+\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\end{aligned} \tag{5.115}$$

with the bounds

$$\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\le\frac{\theta\lambda M^2\acute{d}}{4}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n-\theta)}\right\|^2, \tag{5.116}$$

$$\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)\leq\frac{\delta\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\frac{\delta\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{(n-\theta)}\right\|^2,\tag{5.117}$$

$$\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{(n+\tilde{\theta})}\right)\leq\frac{\theta\lambda M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2,\tag{5.118}$$

and

$$\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)\leq\frac{\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\frac{\delta^2\theta\lambda M^2\acute{d}}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2.\tag{5.119}$$

For the first $g_a(\cdot,\cdot)$ term

$$\begin{aligned}(1-2\theta)\,\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}_{\delta^{(n+\tilde{\theta})}}^{(n+\tilde{\theta})}\right)&=(1-2\theta)\,\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\+(1-2\theta)\,\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)&+(1-2\theta)\,\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&+(1-2\theta)\,\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right),\end{aligned}\tag{5.120}$$

with each part bounded by

$$(1-2\theta)\,\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\leq(1-2\theta)\,\lambda 4M\acute{d}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2,\tag{5.121}$$

$$\begin{aligned}&(1-2\theta)\,\lambda\left(g_a(\boldsymbol{F}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&\qquad\leq4\,(1-2\theta)\,\lambda\acute{d}^4M^4\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\delta^2\,(1-2\theta)\,\lambda\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2,\end{aligned}\tag{5.122}$$

$$\begin{aligned}&(1-2\theta)\,\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&\qquad\leq4\,(1-2\theta)\,\lambda M^2\acute{d}^2\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2+(1-2\theta)\,\lambda\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2,\end{aligned}\tag{5.123}$$

and

$$\begin{aligned}&(1-2\theta)\,\lambda\left(g_a(\boldsymbol{\Gamma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})}),\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&\qquad\leq4\,(1-2\theta)\,\lambda\acute{d}^4M^4\left\|\boldsymbol{\Gamma}^{(n+\tilde{\theta})}\right\|^2+\delta^2\,(1-2\theta)\,\lambda\left\|\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2.\end{aligned}\tag{5.124}$$

Next,

$$\begin{aligned}\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}_{\delta^{(n)}}^{(n+\tilde{\theta})}\right)&=\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&+\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right)+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\\&+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right),\end{aligned}\tag{5.125}$$

and has the bounds

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\tilde{\theta})}\right)\leq4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\theta\lambda\left\|\boldsymbol{F}^{(n)}\right\|^2,\tag{5.126}$$

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{F}^{(n)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.127}$$

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2, \tag{5.128}$$

and

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \tag{5.129}$$

The last term in this grouping is

$$\begin{aligned}\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}_{\delta^{(n-\theta)}}^{\left(n+\tilde{\theta}\right)}\right)&=\theta\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&+\theta\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\\&+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right),\end{aligned} \tag{5.130}$$

and we use the bounds

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2, \tag{5.131}$$

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{5.132}$$

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\lambda\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2, \tag{5.133}$$

and

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \tag{5.134}$$

5.1.3 Bounding $\mathcal{B}_3^n(\boldsymbol{\sigma},\boldsymbol{\tau})-\mathcal{B}_3^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau})$

In this sub section we determine bounds on the terms in expression (5.21), $\mathcal{B}_3^n(\boldsymbol{\sigma},\boldsymbol{\tau})-\mathcal{B}_3^n(\boldsymbol{\sigma}_h,\boldsymbol{\tau})$, for $\boldsymbol{\tau}=\boldsymbol{F}^{(n+\theta)}$. Thus,

$$\begin{aligned}\mathcal{B}_3^n(\boldsymbol{\sigma},\boldsymbol{F}^{(n+\theta)})-\mathcal{B}_3^n(\boldsymbol{\sigma}_h,\boldsymbol{F}^{(n+\theta)}):=\;&\lambda\left(d_te_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)+\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)\\&+(1-2\theta)\,\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta-1)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)+\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)+\tilde{\theta}\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)\\&+\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)\\&+\tilde{\theta}\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)+\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right).\end{aligned} \tag{5.135}$$

Each of the terms in expression (5.135) are now bounded. Here

$$\lambda\left(d_t\left(e_{\boldsymbol{\sigma}}^{(n+\theta)}\right),\boldsymbol{F}^{(n+\theta)}\right)=\lambda\left(d_t\left(\boldsymbol{F}^{(n+\theta)}\right),\boldsymbol{F}^{(n+\theta)}\right)+\lambda\left(d_t\left(\boldsymbol{\Gamma}^{(n+\theta)}\right),\boldsymbol{F}^{(n+\theta)}\right), \tag{5.136}$$

with

$$\lambda\left(d_t\left(\boldsymbol{F}^{(n+\theta)}\right),\boldsymbol{F}^{(n+\theta)}\right)\geq\frac{\lambda}{2\Delta t}\left(\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2-\left\|\boldsymbol{F}^{(n+\theta)-1}\right\|^2\right), \tag{5.137}$$

and

$$\lambda\left(d_t\left(\boldsymbol{\Gamma}^{(n+\theta)}\right),\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\lambda}{4}\left\|d_t\left(\boldsymbol{\Gamma}^{(n+\theta)}\right)\right\|^2+\lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2. \tag{5.138}$$

Next,

$$\begin{aligned}\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)&=\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).\end{aligned} \tag{5.139}$$

Bounding each term we have

$$\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\boldsymbol{F}^{(n+\theta)}\right)=\theta\omega\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.140}$$

$$\theta\omega\left(\boldsymbol{F}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.141}$$

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.142}$$

and

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n+\theta)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n+\theta)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2. \tag{5.143}$$

Again, expanding the term

$$\begin{aligned}\theta\omega\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)&=\theta\omega\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\omega\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).\end{aligned} \tag{5.144}$$

Bounding each of these

$$\theta\omega\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.145}$$

$$\theta\omega\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.146}$$

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega}{2}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\frac{\theta\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.147}$$

and

$$\theta\omega\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\omega M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2. \tag{5.148}$$

Next

$$\begin{aligned}(1-2\theta)\,\omega\left(e_{\boldsymbol{\sigma}}^{(n+\theta-1)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)&=(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta-1)},\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta-1)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)+(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta-1)},\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta-1)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right),\end{aligned} \tag{5.149}$$

and has bounds

$$(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta-1)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+\theta-1)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.150}$$

$$
\begin{aligned}
&(1-2\theta)\,\omega\left(\boldsymbol{F}^{(n+\theta-1)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\
&\qquad\leq\frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n+\theta-1)}\right\|^2+\delta^2\,(1-2\theta)\,\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2,
\end{aligned}
\tag{5.151}
$$

$$
(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta-1)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{\Gamma}^{(n+\theta-1)}\right\|^2+\frac{(1-2\theta)\,\omega}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.152}
$$

and

$$
\begin{aligned}
&(1-2\theta)\,\omega\left(\boldsymbol{\Gamma}^{(n+\theta-1)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\
&\qquad\leq\frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n+\theta-1)}\right\|^2+\delta^2\,(1-2\theta)\,\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2.
\end{aligned}
\tag{5.153}
$$

Splitting

$$
\begin{aligned}
\theta\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)&=\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\
&\quad+\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right),
\end{aligned}
\tag{5.154}
$$

we have

$$
\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n)}\right\|^2+\frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.155}
$$

$$
\theta\tilde{\omega}\left(\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n)}\right\|^2+\delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.156}
$$

$$
\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\frac{\theta\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.157}
$$

and

$$
\theta\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2.
\tag{5.158}
$$

The term

$$
\begin{aligned}
\tilde{\theta}\tilde{\omega}\left(e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)&=\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)+\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\
&\quad+\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)+\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).
\end{aligned}
\tag{5.159}
$$

Here each term is treated in the following manner.

$$
\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.160}
$$

$$
\tilde{\theta}\tilde{\omega}\left(\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\delta^2\tilde{\theta}\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.161}
$$

$$
\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2+\frac{\tilde{\theta}\tilde{\omega}}{2}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\tag{5.162}
$$

and

$$
\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4}\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2+\delta^2\tilde{\theta}\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2.
\tag{5.163}
$$

Examining the convective terms,

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n-\theta)},\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)=\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)$$
$$+\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)+\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)$$
$$+\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).\quad(5.164)$$

Here each part is bounded

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n-\theta)}\right\|^2,\quad(5.165)$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\delta\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n-\theta)}\right\|^2+\frac{\delta\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2,\quad(5.166)$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\tilde{\theta}\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2,\quad(5.167)$$

and

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\Gamma}^{(n-\theta)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2+\frac{\delta^2\tilde{\theta}\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2.\quad(5.168)$$

The second convective term is:

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla e_{\boldsymbol{\sigma}}^{(n)},\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)=\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)+\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right),\quad(5.169)$$

with bounds given by

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2,\quad(5.170)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\delta\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2+\frac{\delta\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2,\quad(5.171)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\lambda\acute{d}M^2}{4}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2,\quad(5.172)$$

and

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\Gamma}^{(n)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2+\frac{\delta^2\theta\lambda\acute{d}M^2}{2}\left\|\nabla\boldsymbol{\Gamma}^{(n)}\right\|^2.\quad(5.173)$$

Lastly the $g_a(\cdot,\cdot)$ terms are considered. Specifically

$$\tilde{\theta}\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}_{\delta^{(n-\theta)}}^{(n+\theta)}\right)=\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right)$$
$$+\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)+\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right)$$
$$+\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).\quad(5.174)$$

Handling each part

$$\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right)\leq 4\tilde{\theta}\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\tilde{\theta}\lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.175}$$

$$\tilde{\theta}\lambda\left(g_a(\boldsymbol{F}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq 4\tilde{\theta}\lambda M^4\acute{d}^4\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\delta^2\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.176}$$

$$\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right)\leq 4\tilde{\theta}\lambda M^2\acute{d}^2\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2+\tilde{\theta}\lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.177}$$

and

$$\tilde{\theta}\lambda\left(g_a(\boldsymbol{\Gamma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq 4\tilde{\theta}\lambda M^4\acute{d}^4\left\|\boldsymbol{\Gamma}^{(n-\theta)}\right\|^2+\delta^2\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2. \tag{5.178}$$

For the last term:

$$\begin{aligned}\theta\lambda\left(g_a(e_{\boldsymbol{\sigma}}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}_{\delta^{(n)}}^{(n+\theta)}\right)&=\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\\&\quad+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\theta)}\right)+\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right),\end{aligned} \tag{5.179}$$

and each term is bounded by

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\theta)}\right)\leq 4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{(n)}\right\|^2+\theta\lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.180}$$

$$\theta\lambda\left(g_a(\boldsymbol{F}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.181}$$

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\boldsymbol{F}^{(n+\theta)}\right)\leq 4\theta\lambda M^2\acute{d}^2\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2+\theta\lambda\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2, \tag{5.182}$$

and

$$\theta\lambda\left(g_a(\boldsymbol{\Gamma}^{(n)},\nabla\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq 4\theta\lambda M^4\acute{d}^4\left\|\boldsymbol{\Gamma}^{(n)}\right\|^2+\delta^2\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2. \tag{5.183}$$

5.1.4 Bound the $\mathcal{I}_1^n(\boldsymbol{F}^{(n+1)})$ Terms in (5.7)

The bounds on the $\mathcal{I}_1^n(\boldsymbol{\tau})$ term are now found with $\boldsymbol{\tau}=\boldsymbol{F}^{(n+1)}$. This term is broken into two pieces in the following manner:

$$\mathcal{I}_1^n(\boldsymbol{F}^{(n+1)})=Int_{\boldsymbol{\sigma}_1 g}(\boldsymbol{F}^{(n+1)})+Int_{\boldsymbol{\sigma}_1 up}(\boldsymbol{F}^{(n+1)}),$$

with

$$\begin{aligned}Int_{\boldsymbol{\sigma}_1 g}(\boldsymbol{F}^{(n+1)})&=\lambda\left(d_t\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})},\boldsymbol{F}^{(n+1)}\right)+\ \omega\left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)}+\theta\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\boldsymbol{F}^{(n+1)}\right)\\&\quad+\tilde{\omega}\left(\theta\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\boldsymbol{F}^{(n+1)}\right)\\&\quad+\lambda\left(\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})},\boldsymbol{F}^{(n+1)}\right)\\&\quad+\lambda\left(\theta g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)})+\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})-g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})},\nabla\boldsymbol{u}^{(n+\frac{1}{2})}),\boldsymbol{F}^{(n+1)}\right)\\&\quad+2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right),\end{aligned}$$

and

$$
\begin{aligned}
Int_{\boldsymbol{\sigma}_{1up}}(\boldsymbol{F}^{(n+1)}) = {} & \lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\lambda\tilde{\theta}\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\omega\theta\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\omega\theta\left(\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\omega\left(1-2\theta\right)\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\tilde{\omega}\theta\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\tilde{\omega}\tilde{\theta}\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla\boldsymbol{u}^{(n+\frac{1}{2})}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla\boldsymbol{u}^{(n+\frac{1}{2})}), \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right) \\
& +\tilde{\theta} 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}), \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right).
\end{aligned}
$$

Bounding each of the $Int_{\boldsymbol{\sigma}_{1g}}(\boldsymbol{F}^{(n+1)})$ terms:

$$
\lambda\left(d_t\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})}, \boldsymbol{F}^{(n+1)}\right) \le \frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2})}\right\|^2 + \lambda\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.184}
$$

$$
\begin{aligned}
\omega\left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{F}^{(n+1)}\right) \quad \le \quad & \frac{\omega}{4}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2 && (5.185) \\
& + \omega\left\|\boldsymbol{F}^{(n+1)}\right\|^2, && (5.186)
\end{aligned}
$$

$$
\tilde{\omega}\left(\theta\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{F}^{(n+1)}\right) \le \frac{\tilde{\omega}}{4}\left\|\theta\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2 + \tilde{\omega}\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \tag{5.187}
$$

$$
\begin{aligned}
& \lambda\left(\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \boldsymbol{F}^{(n+1)}\right) \\
& \qquad \le \frac{\lambda}{4}\left\|\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2 + \lambda\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \qquad (5.188)
\end{aligned}
$$

$$
\begin{aligned}
& \lambda\left(\theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla\boldsymbol{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla\boldsymbol{u}^{(n+\frac{1}{2})}), \boldsymbol{F}^{(n+1)}\right) \\
& \qquad \le \frac{\lambda}{4}\left\|\theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla\boldsymbol{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla\boldsymbol{u}^{(n+\frac{1}{2})})\right\|^2 \\
& \qquad\qquad + \lambda\left\|\boldsymbol{F}^{(n+1)}\right\|^2, \qquad (5.189)
\end{aligned}
$$

and

$$2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}),\boldsymbol{F}^{(n+1)}\right)$$
$$\leq\frac{\alpha}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})})\right\|^2+2\alpha\left\|\boldsymbol{F}^{(n+1)}\right\|^2. \quad (5.190)$$

Bounding each of the $Int_{\boldsymbol{\sigma}_{1up}}(\boldsymbol{F}^{(n+1)})$ terms:

$$\lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\lambda\theta\delta^2M^2\acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}\right\|^2+\lambda\theta\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.191)$$

$$\lambda\tilde{\theta}\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\lambda\tilde{\theta}\delta^2M^2\acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{(n+\frac{1}{2})}\right\|^2+\lambda\tilde{\theta}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.192)$$

$$\omega\theta\left(\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\omega\theta\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2$$
$$+\omega\theta\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.193)$$

$$\omega\theta\left(\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\omega\theta\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2$$
$$+\omega\theta\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.194)$$

$$\omega\left(1-2\theta\right)\left(\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\omega\left(1-2\theta\right)\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2$$
$$+\omega\left(1-2\theta\right)\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.195)$$

$$\tilde{\omega}\theta\left(\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\tilde{\omega}\theta\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2+\tilde{\omega}\theta\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.196)$$

$$\tilde{\omega}\tilde{\theta}\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)\leq\frac{\tilde{\omega}\tilde{\theta}\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2$$
$$+\tilde{\omega}\tilde{\theta}\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.197)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)$$
$$\leq\frac{\theta\lambda\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.198)$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})},\delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+1)}\right)$$
$$\leq\frac{\tilde{\theta}\lambda\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2+\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2, \quad (5.199)$$

$$\theta\lambda \left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}), \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right)$$
$$\leq \frac{\theta\lambda\delta^2 M^2 \acute{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) \right\|^2 + \theta\lambda \left\| \nabla \boldsymbol{F}^{(n+1)} \right\|^2, \quad (5.200)$$

$$\tilde{\theta}\lambda \left(g_a(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}, \nabla \boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}), \delta \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right)$$
$$\leq \frac{\tilde{\theta}\lambda\delta^2 M^2 \acute{d}}{4} \left\| g_a(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}, \nabla \boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) \right\|^2 + \tilde{\theta}\lambda \left\| \nabla \boldsymbol{F}^{(n+1)} \right\|^2, \quad (5.201)$$

$$\theta 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right)$$
$$\leq \frac{\theta\alpha\delta^2 M^2 \acute{d}}{2} \left\| \mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n)}) \right\|^2 + \theta 2\alpha \left\| \nabla \boldsymbol{F}^{(n+1)} \right\|^2, \quad (5.202)$$

$$\tilde{\theta} 2\alpha \left(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}), \delta \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{F}^{(n+1)} \right)$$
$$\leq \frac{\tilde{\theta}\alpha\delta^2 M^2 \acute{d}}{2} \left\| \mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) \right\|^2 + \tilde{\theta} 2\alpha \left\| \nabla \boldsymbol{F}^{(n+1)} \right\|^2. \quad (5.203)$$

5.1.5 Bounding the $\mathcal{I}_2^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)})$ Terms in (5.13)

Here we bound the terms in $\mathcal{I}_2^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)})$ in expression (5.13) in two phases, first handling the standard Galerkin terms, and then the upwinded terms. Consider

$$\mathcal{I}_2^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}) = \mathcal{I}_{2\,g}^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}) + \mathcal{I}_{2\,up}^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}), \quad (5.204)$$

where

$$\begin{aligned}
\mathcal{I}_{2\,g}^n(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}) \quad = \quad & \lambda \left(d_t \boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right) \\
& + \omega \left(\theta \boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\, \boldsymbol{\sigma}^{(n+\theta)} + \theta \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right) \\
& + \tilde{\omega} \left((1-2\theta)\, \boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} + \theta \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right) \\
& + \lambda \Big((1-2\theta)\, \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} + \theta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} \\
& \qquad - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big) \\
& + \lambda \Big((1-2\theta)\, g_a(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}, \nabla \boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) \\
& \qquad + \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}), \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big) \\
& + 2\alpha \Big(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}) - \theta \mathbf{d}(\boldsymbol{u}^{(n)}) - (1-2\theta)\, \mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) \\
& \qquad - \theta \mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big),
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{I}^n_{2\,up}(\boldsymbol{F}^{(n+\tilde{\theta})}) &= \lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \lambda\,(1-2\theta)\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \lambda\theta\left(-\boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) + \theta\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ (1-2\theta)\,\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\omega\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ (1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ (1-2\theta)\,\lambda\left(\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{(n+\frac{1}{2}-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ (1-2\theta)\,\lambda\left(g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla\boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}), \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n)}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla\boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}), \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ (1-2\theta)\,2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}), \delta\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&+ \theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right).
\end{aligned}
$$

Bounding the $\mathcal{I}^n_{2\,g}(\boldsymbol{F}^{(n+\tilde{\theta})})$ terms:

$$
\lambda\left(d_t\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \le \frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{\sigma}_t^{(n+\frac{1}{2}-\theta)}\right\|^2 + \lambda\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2, \tag{5.205}
$$

$$
\begin{aligned}
&\omega\left(\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&\qquad \le \frac{\omega}{4}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}\right\|^2 + \omega\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2,
\end{aligned} \tag{5.206}
$$

$$
\begin{aligned}
&\tilde{\omega}\left((1-2\theta)\,\boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta\boldsymbol{\sigma}^{(n)} + \theta\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \boldsymbol{F}^{(n+\tilde{\theta})}\right) \\
&\qquad \le \frac{\tilde{\omega}}{4}\left\|(1-2\theta)\,\boldsymbol{\sigma}^{(n+\tilde{\theta})} + \theta\boldsymbol{\sigma}^{(n)} + \theta\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}\right\|^2 + \tilde{\omega}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2,
\end{aligned} \tag{5.207}
$$

$$\lambda\Big((1-2\theta)\, \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} + \theta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big)$$
$$\leq \frac{\lambda}{4} \left\| (1-2\theta)\, \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} + \theta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \theta \boldsymbol{u}^{(n-\theta)} \cdot \nabla \boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)} \cdot \nabla \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2$$
$$+ \lambda \left\| \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.208)$$

$$\lambda\Big((1-2\theta)\, g_a(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}, \nabla \boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)})$$
$$+ \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}), \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big)$$
$$\leq \frac{\lambda}{4} \Big\| (1-2\theta)\, g_a(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}, \nabla \boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) + \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)})$$
$$+ \theta g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \nabla \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}) \Big\|^2 + \lambda \left\| \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.209)$$

and

$$2\alpha\Big(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}) - \theta \mathbf{d}(\boldsymbol{u}^{(n)}) - (1-2\theta)\, \mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) - \theta \mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \Big)$$
$$\leq \frac{\alpha}{2} \left\| \mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}) - \theta \mathbf{d}(\boldsymbol{u}^{(n)}) - (1-2\theta)\, \mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}) - \theta \mathbf{d}(\boldsymbol{u}^{(n-\theta)}) \right\|^2 + 2\alpha \left\| \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2. \quad (5.210)$$

Bounding the terms in $\mathcal{I}^n_{2\,up}(\boldsymbol{F}^{\left(n+\tilde{\theta}\right)})$

$$\lambda\theta \left(-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right) \leq \frac{\lambda\theta\delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 + \lambda\theta \left\| \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.211)$$

$$\lambda\,(1-2\theta) \left(-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}, \delta \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right)$$
$$\leq \frac{\lambda\,(1-2\theta)\,\delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 + \lambda\,(1-2\theta) \left\| \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.212)$$

$$\lambda\theta \left(-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}, \delta \boldsymbol{u}^{(n-\theta)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right) \leq \frac{\lambda\theta\delta^2 M^2 \acute{d}}{4} \left\| -\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 + \lambda\theta \left\| \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.213)$$

$$\theta\omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \delta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right)$$
$$\leq \frac{\theta\omega\delta^2 M^2 \acute{d}}{4} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 + \theta\omega \left\| \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.214)$$

$$(1-2\theta)\,\omega \left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \delta \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right)$$
$$\leq \frac{(1-2\theta)\,\omega\delta^2}{4\epsilon_1} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 + (1-2\theta)\,\omega\epsilon_1 \left\| \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} \cdot \nabla \boldsymbol{F}^{\left(n+\tilde{\theta}\right)} \right\|^2, \quad (5.215)$$

$$\theta\omega\left(\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\omega\delta^{2}M^{2}\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\theta\omega\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.216)$$

$$(1-2\theta)\,\tilde{\omega}\left(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\tilde{\omega}}{4\epsilon_{1}}\left\|\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\delta^{2}\,(1-2\theta)\,\tilde{\omega}\epsilon_{1}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.217)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\tilde{\omega}\delta^{2}M^{2}\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.218)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\tilde{\omega}\delta^{2}M^{2}\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.219)$$

$$(1-2\theta)\,\lambda\left(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\lambda}{4\epsilon_{1}}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}$$
$$+\delta^{2}\,(1-2\theta)\,\lambda\epsilon_{1}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.220)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\lambda\delta^{2}M^{2}\acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.221)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\lambda\delta^{2}M^{2}\acute{d}}{4}\left\|\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^{2}+\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.222)$$

$$(1-2\theta)\,\lambda\left(g_{a}(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)},\nabla\boldsymbol{u}^{\left(n+\tilde{\theta}\right)})-g_{a}(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}),\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\lambda}{4\epsilon_{1}}\left\|g_{a}(\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)},\nabla\boldsymbol{u}^{\left(n+\tilde{\theta}\right)})-g_{a}(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})\right\|^{2}$$
$$+\delta^{2}\,(1-2\theta)\,\lambda\epsilon_{1}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^{2},\quad(5.223)$$

$$\theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\lambda\delta^2M^2\acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.224)$$

$$\theta\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\lambda\delta^2M^2\acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.225)$$

$$\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta\alpha\delta^2M^2\acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{(n)})\right\|^2+\theta 2\alpha\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.226)$$

$$(1-2\theta)\,2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}),\delta\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{(1-2\theta)\,\alpha}{2\epsilon_1}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{\left(n+\tilde{\theta}\right)})\right\|^2+\delta^2\,(1-2\theta)\,2\alpha\epsilon_1\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2, \quad (5.227)$$

and

$$\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right)$$
$$\leq\frac{\theta 2\alpha\delta^2M^2\acute{d}}{4}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2+\theta 2\alpha\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2. \quad (5.228)$$

5.1.6 Bounding the $\mathcal{I}_3^n(\boldsymbol{F}^{(n+\theta)})$ Terms in (5.20)

Bounding the $\mathcal{I}_3^n(\boldsymbol{F}^{(n+\theta)})$ terms in (5.20) is done in two phases, first the standard Galerkin terms and then the upwinded terms.

$$\mathcal{I}_3^n(\boldsymbol{F}^{(n+\theta)})=\mathcal{I}_{3\,g}^n(\boldsymbol{F}^{(n+\theta)})+\mathcal{I}_{3\,up}^n(\boldsymbol{F}^{(n+\theta)}), \quad (5.229)$$

where

$$\begin{aligned}\mathcal{I}_{3\,g}^n(\boldsymbol{F}^{(n+\theta)})&=\lambda\left(d_t\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)},\boldsymbol{F}^{(n+\theta)}\right)\\&+\tilde{\omega}\left(\theta\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\boldsymbol{F}^{(n+\theta)}\right)\\&+\omega\left(\theta\boldsymbol{\sigma}^{(n+\theta)}+\theta\boldsymbol{\sigma}^{(n)}+(1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\boldsymbol{F}^{(n+\theta)}\right)\\&+\lambda\left(\tilde{\theta}\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}+\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\boldsymbol{F}^{(n+\theta)}\right)\\&+\lambda\left(\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})+\theta g_a(\boldsymbol{\sigma}^{n},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}),\boldsymbol{F}^{(n+\theta)}\right)\\&+2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right),\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{I}^n_{3\,up}(\boldsymbol{F}^{(n+\theta)}) = {} & \lambda\theta\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) + \ \lambda\tilde{\theta}\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta\omega\left(\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta\omega\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \ (1-2\theta)\,\omega\left(\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \tilde{\theta}\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla\boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta\lambda\left(g_a(\boldsymbol{\sigma}^{n}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n)}), \delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right) \\
& + \tilde{\theta} 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)}), \delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right).
\end{aligned}
$$

Bounding the terms in $\mathcal{I}^n_{3\,g}(\boldsymbol{F}^{(n+\theta)})$:

$$
\lambda\left(d_t\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{F}^{(n+\theta)}\right) \le \frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, \tag{5.230}
$$

$$
\begin{aligned}
\tilde{\omega}\left(\theta\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{F}^{(n+\theta)}\right) \ \le \ & \frac{\tilde{\omega}}{4}\left\|\theta\boldsymbol{\sigma}^{(n)} + \tilde{\theta}\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 && (5.231) \\
& + \tilde{\omega}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2, && (5.232)
\end{aligned}
$$

$$
\begin{aligned}
& \omega\left(\theta\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{F}^{(n+\theta)}\right) \\
& \qquad \le \frac{\omega}{4}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \omega\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\end{aligned}
\tag{5.233}
$$

$$
\begin{aligned}
& \lambda\left(\tilde{\theta}\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} + \theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{F}^{(n+\theta)}\right) \\
& \quad \le \frac{\lambda}{4}\left\|\tilde{\theta}\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} + \theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\end{aligned}
\tag{5.234}
$$

$$
\begin{aligned}
& \lambda\left(\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla\boldsymbol{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{n}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \boldsymbol{F}^{(n+\theta)}\right) \\
& \qquad \le \frac{\lambda}{4}\left\|\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla\boldsymbol{u}^{(n-\theta)}) + \theta g_a(\boldsymbol{\sigma}^{n}, \nabla\boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2 \\
& \qquad\qquad + \lambda\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2,
\end{aligned}
\tag{5.235}
$$

$$2\alpha\left(\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\alpha}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2+2\alpha\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2. \quad (5.236)$$

Bounding the $\mathcal{I}^n_{3\,up}(\boldsymbol{F}^{(n+\theta)})$ terms:

$$\lambda\theta\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\lambda\theta\delta^2M^2\acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\lambda\theta\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.237)$$

$$\lambda\tilde{\theta}\left(-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)\leq\frac{\lambda\tilde{\theta}\delta^2M^2\acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\lambda\tilde{\theta}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.238)$$

$$\theta\omega\left(\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\omega\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.239)$$

$$\theta\omega\left(\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\omega\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\theta\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.240)$$

$$(1-2\theta)\,\omega\left(\boldsymbol{\sigma}^{(n+\theta-1)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{(1-2\theta)\,\omega\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta-1)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+(1-2\theta)\,\omega\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.241)$$

$$\theta\tilde{\omega}\left(\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\tilde{\omega}\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\theta\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.242)$$

$$\tilde{\theta}\tilde{\omega}\left(\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\tilde{\theta}\tilde{\omega}\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\tilde{\theta}\tilde{\omega}\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.243)$$

$$\tilde{\theta}\lambda\left(\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\tilde{\theta}\lambda\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.244)$$

$$\theta\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\lambda\delta^2M^2\acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.245)$$

$$\tilde{\theta}\lambda\left(g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\tilde{\theta}\lambda\delta^2M^2\acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2+\tilde{\theta}\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.246)$$

$$\theta\lambda\left(g_a(\boldsymbol{\sigma}^{n},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\lambda\delta^2M^2\acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{n},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2+\theta\lambda\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.247)$$

$$\theta 2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\mathbf{d}(\boldsymbol{u}^{(n)}),\delta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\theta\alpha\delta^2M^2\acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\mathbf{d}(\boldsymbol{u}^{(n)})\right\|^2+\theta 2\alpha\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2, \quad (5.248)$$

$$\tilde{\theta}2\alpha\left(\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\mathbf{d}(\boldsymbol{u}^{(n-\theta)}),\delta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{F}^{(n+\theta)}\right)$$
$$\leq\frac{\tilde{\theta}\alpha\delta^2M^2\acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2+\tilde{\theta}2\alpha\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2. \quad (5.249)$$

5.1.7 Bring all bounds together:

The goal is to now bring all the bounded expressions together into a single expression so that the discrete Gronwall lemma may be applied, and the error estimate established. Thus, bringing all the bounds on the three linear combinations (4.16), (4.17), and (4.18) together into a single expression we obtain

$$
\begin{aligned}
&\frac{\lambda}{2\Delta t}\left(\left\|\boldsymbol{F}^{(n+1)}\right\|^2-\left\|\boldsymbol{F}^{(n)}\right\|^2+\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2-\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2+\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2-\left\|\boldsymbol{F}^{(n+\theta)-1}\right\|^2\right)\\
&+\theta\omega\left\|\boldsymbol{F}^{(n+1)}\right\|^2+(1-2\theta)\,\tilde{\omega}\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2+\theta\omega\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2\\
&+(1-2\theta)\left(\delta\lambda-\epsilon_1\delta^2\left(2+3\lambda+2\alpha\right)\right)\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2\\
\leq\quad&\left\|\boldsymbol{F}^{(n)}\right\|^2\left(\theta\lambda+\theta\omega+\frac{3\theta\tilde{\omega}}{2}+4\theta\lambda M\acute{d}+4\theta\lambda M^2\acute{d}^2+8\theta\lambda M^4\acute{d}^4+\frac{\theta\omega M^2\acute{d}}{2}+\frac{3\theta\tilde{\omega}M^2\acute{d}}{4}\right)\\
&+\left\|\boldsymbol{F}^{(n+1)}\right\|^2\left(1+4\lambda+2\alpha+\theta+\tilde{\theta}\tilde{\omega}+(1-2\theta)\,\omega+\frac{\theta\omega}{2}+\frac{\theta\omega M^2\acute{d}}{4}+\frac{\lambda\acute{d}M^2}{2}+2\lambda\acute{d}M\right)\\
&+\left\|\boldsymbol{F}^{(n+\theta-1)}\right\|^2\left(\frac{(1-2\theta)\,\omega}{2}\left(1+\frac{M^2\acute{d}}{2}\right)\right)\\
&+\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2\Bigg(1+5\lambda+\theta\omega+\tilde{\omega}+\tilde{\theta}\lambda+2\alpha+\frac{\theta\omega M^2\acute{d}}{2}+(1-2\theta)\,\omega+\frac{\tilde{\theta}\omega}{2}+\frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\\
&\qquad+\frac{\theta\omega M^2\acute{d}}{4}+\frac{\omega}{2}+\frac{\lambda\acute{d}M^2}{2}+4\theta\lambda M^4\acute{d}^4+4\theta\lambda M^2\acute{d}^2+\frac{(1-2\theta)\,\omega}{4\epsilon_1}\Bigg)\\
&+\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2\left(\frac{\tilde{\omega}}{2}+\theta\lambda+\frac{\tilde{\omega}M^2\acute{d}}{4}+4\tilde{\theta}\lambda M^2\acute{d}^2+4\lambda M^4\acute{d}^4\right)\\
&+\left\|\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2\Bigg(1+4\lambda+\omega+\theta+\theta(1-2\omega)+(1-2\theta)\,\lambda+2\alpha+\frac{\tilde{\theta}\tilde{\omega}}{2}+\frac{(1-2\theta)\,\tilde{\omega}}{2}+\frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4}\\
&\qquad+\frac{(1-2\theta)\,\lambda M^2\acute{d}}{4}+4\tilde{\theta}\lambda\acute{d}M+\theta\lambda M^2\acute{d}+4\,(1-2\theta)\,\lambda M\acute{d}\\
&\qquad+4\,(1-2\theta)\,\lambda M^4\acute{d}^4+4\tilde{\theta}\lambda M^4\acute{d}^4+16\theta\lambda M^2\acute{d}^2+\frac{(1-2\theta)\,\tilde{\omega}}{4\epsilon_1}\Bigg)\\
&+\left\|\nabla\boldsymbol{F}^{(n-\theta)}\right\|^2\left(\lambda+\frac{\delta\lambda M^2\acute{d}}{2}+\delta^2\tilde{\theta}\lambda\right)+\left\|\nabla\boldsymbol{F}^{(n)}\right\|^2\left(3\lambda\theta\left(1+\frac{\delta\acute{d}M^2}{2}\right)\right)\\
&+\left\|\nabla\boldsymbol{F}^{\left(n+\tilde{\theta}\right)}\right\|^2\left(2\lambda+2\theta+3\theta\lambda+4\alpha\theta+\frac{\lambda M^2\acute{d}}{2}+\frac{\theta\lambda M^2\acute{d}}{2}+\theta\delta\lambda M^2\acute{d}+2\delta^2\left(\lambda+2\theta\right)\right)\\
&+\left\|\nabla\boldsymbol{F}^{(n+\theta)}\right\|^2\left(1+3\lambda+2\alpha+\frac{\lambda\acute{d}M^2}{2}+\frac{\delta\lambda\acute{d}M^2}{2}+\delta^2\left(\omega+2\theta+2\lambda+(1-2\theta)\,\omega+2\tilde{\theta}\tilde{\omega}\right)\right)\\
&+\left\|\nabla\boldsymbol{F}^{(n+1)}\right\|^2\left(1+3\lambda+2\alpha+\frac{\delta\lambda\acute{d}M^2}{2}+\frac{\lambda\acute{d}M^2}{2}+\delta^2\omega\left(2-\theta\right)+2\delta^2\tilde{\omega}+2\delta^2\lambda\right)\\
&+H_{\boldsymbol{\Gamma}}+H_g+\delta^2H_{upA}+\delta^2H_{upB}.
\end{aligned}
\tag{5.250}
$$

Here the terms $H_{\boldsymbol{\Gamma}}, H_g, H_{upA}$, and H_{upB} in (5.250) are defined by

$$
\begin{aligned}
H_g \quad := \quad & \frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)}\right\|^2+\frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
& +\frac{\lambda}{4}\left\|d_t\boldsymbol{\sigma}^{(n+\theta)}-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2+\frac{\omega}{4}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)}+\theta\boldsymbol{\sigma}^{(n+1)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\omega}{4}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)}+(1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta)}+\theta\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
& +\frac{\omega}{4}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)}+\theta\boldsymbol{\sigma}^{(n)}+(1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\tilde{\omega}}{4}\left\|\theta\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\tilde{\omega}}{4}\left\|(1-2\theta)\,\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}+\theta\boldsymbol{\sigma}^{(n)}+\theta\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
& +\frac{\tilde{\omega}}{4}\left\|\theta\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{\sigma}^{(n-\theta)}-\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\lambda}{4}\left\|\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}+\tilde{\theta}\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}-\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\lambda}{4}\left\|(1-2\theta)\,\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)}+\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}\right. \\
& \qquad\left.+\theta\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}-\ \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
& +\frac{\lambda}{4}\left\|\tilde{\theta}\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)}+\theta\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)}-\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
& +\frac{\lambda}{4}\left\|\theta g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)})+\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}\right)})\right\|^2 \\
& +\frac{\lambda}{4}\left\|\,(1-2\theta)\,g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})+\theta g_a(\boldsymbol{\sigma}^{(n)},\nabla\boldsymbol{u}^{(n)})\right. \\
& \qquad\left.+\,\theta g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})-g_a(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)},\nabla\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})\right\|^2 \\
& +\frac{\lambda}{4}\left\|\tilde{\theta}g_a(\boldsymbol{\sigma}^{(n-\theta)},\nabla\boldsymbol{u}^{(n-\theta)})+\theta g_a(\boldsymbol{\sigma}^{n},\nabla\boldsymbol{u}^{(n)})-g_a(\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)},\nabla\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2 \\
& +\frac{\alpha}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})})\right\|^2 \\
& +\frac{\alpha}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-(1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})})-\theta\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2 \\
& +\frac{\alpha}{2}\left\|\mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})-\theta\mathbf{d}(\boldsymbol{u}^{(n)})-\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2,
\end{aligned} \tag{5.251}
$$

$$
\begin{aligned}
H_{upA} \;=\;& \frac{\lambda M^2 \acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)}\right\|^2 + \frac{\lambda M^2 \acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \frac{\lambda M^2 \acute{d}}{4}\left\|-\boldsymbol{\sigma}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\omega\theta M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 + \frac{\omega\theta M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\omega\left(1-2\theta\right) M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\theta\omega M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \frac{\tilde{\omega}\theta M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\tilde{\omega}\tilde{\theta} M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 + \frac{\left(1-2\theta\right)\omega}{4\epsilon_1}\left\|\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\left(1-2\theta\right)\tilde{\omega}}{4\epsilon_1}\left\|\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \frac{\theta\omega M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\theta\tilde{\omega} M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \frac{\theta\tilde{\omega} M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\theta\omega M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \frac{\theta\omega M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\left(1-2\theta\right)\omega M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n+\theta-1)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \frac{\theta\tilde{\omega} M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\tilde{\theta}\tilde{\omega} M^2 \acute{d}}{4}\left\|\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\theta\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\tilde{\theta}\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\left(1-2\theta\right)\lambda}{4\epsilon_1}\left\|\boldsymbol{u}^{\left(n+\tilde{\theta}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\theta\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\theta\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \\
&+\frac{\tilde{\theta}\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{(n-\theta)}\cdot\nabla\boldsymbol{\sigma}^{(n-\theta)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \\
&+\frac{\theta\lambda M^2 \acute{d}}{4}\left\|\boldsymbol{u}^{(n)}\cdot\nabla\boldsymbol{\sigma}^{(n)} - \boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}\cdot\nabla\boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2,
\end{aligned}
\tag{5.252}
$$

$$
\begin{aligned}
H_{upB} \;=\; & \frac{\tilde{\theta}\lambda M^2 \acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\frac{1}{2})})\right\|^2 \\
& + \frac{(1-2\theta)\,\lambda}{4\epsilon_1}\left\|g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \boldsymbol{u}^{(n+\frac{1}{2}-\theta)})\right\|^2 \\
& + \frac{\theta\lambda M^2 \acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \boldsymbol{u}^{(n+\frac{1}{2}-\theta)})\right\|^2 \\
& + \frac{\theta\lambda M^2 \acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2}-\theta)}, \nabla \boldsymbol{u}^{(n+\frac{1}{2}-\theta)})\right\|^2 \\
& + \frac{\tilde{\theta}\lambda M^2 \acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{(n-\theta)}, \nabla \boldsymbol{u}^{(n-\theta)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\theta-\frac{1}{2})})\right\|^2 \\
& + \frac{\theta\lambda M^2 \acute{d}}{4}\left\|g_a(\boldsymbol{\sigma}^{n}, \nabla \boldsymbol{u}^{(n)}) - g_a(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\theta-\frac{1}{2})})\right\|^2 \\
& + \frac{\theta\alpha M^2 \acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n)})\right\|^2 + \frac{\tilde{\theta}\alpha M^2 \acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})})\right\|^2 \\
& + \frac{\theta\alpha M^2 \acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n)})\right\|^2 + \frac{(1-2\theta)\,\alpha}{2\epsilon_1}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})})\right\|^2 \\
& + \frac{\theta 2\alpha M^2 \acute{d}}{4}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2}-\theta)}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2 + \frac{\theta\alpha M^2 \acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n)})\right\|^2 \\
& + \frac{\tilde{\theta}\alpha M^2 \acute{d}}{2}\left\|\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})}) - \mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2, \qquad (5.253)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{\mathbf{\Gamma}} \;:=\; & \frac{\lambda}{4}\left\|d_t\left(\mathbf{\Gamma}^{(n+\theta)}\right)\right\|^2 + \frac{\lambda}{4}\left\|d_t\left(\mathbf{\Gamma}^{(n+1)}\right)\right\|^2 + \frac{\lambda}{4}\left\|d_t\left(\mathbf{\Gamma}^{(n+\tilde{\theta})}\right)\right\|^2 \\
& + \left\|\mathbf{\Gamma}^{(n-\theta)}\right\|^2\left(\theta\lambda + \frac{\tilde{\omega}}{2} + \frac{\tilde{\omega}M^2\acute{d}}{4} + 4\tilde{\theta}\lambda M^2\acute{d}^2 + 4\lambda M^4\acute{d}^4\right) \\
& + \left\|\mathbf{\Gamma}^{(n+\theta)}\right\|^2\left(\frac{\omega}{2} + \frac{\theta\omega}{2} + 3\frac{\theta\omega M^2\acute{d}}{4} + \frac{(1-2\theta)\,\omega}{4\epsilon_1} + \frac{(1-2\theta)\,\omega}{2} + \frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\right) \\
& + \left\|\mathbf{\Gamma}^{(n+1)}\right\|^2\left(\frac{\theta\omega}{2}\right) + \left\|\mathbf{\Gamma}^{(n+\theta-1)}\right\|^2\left(\frac{(1-2\theta)\,\omega}{2} + \frac{(1-2\theta)\,\omega M^2\acute{d}}{4}\right) \\
& + \left\|\mathbf{\Gamma}^{(n+\tilde{\theta})}\right\|^2\left(\frac{\tilde{\theta}\tilde{\omega}}{2} + \frac{(1-2\theta)\,\tilde{\omega}}{2} + \frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{4} + \frac{(1-2\theta)\,\tilde{\omega}}{4\epsilon_1}\right. \\
& \qquad \left. + 4\tilde{\theta}\lambda M\acute{d} + 4\,(1-2\theta)\,\lambda M^2\acute{d}^2 + 4\tilde{\theta}\lambda M^4\acute{d}^4 + 4\,(1-2\theta)\,\lambda M^4\acute{d}^4\right) \\
& + \left\|\mathbf{\Gamma}^{(n)}\right\|^2\left(2\theta\lambda + \theta\omega + 3\frac{\theta\tilde{\omega}}{2} + 4\theta\lambda M\acute{d} + 2\frac{\theta\omega M^2\acute{d}}{4} + 3\frac{\theta\tilde{\omega}M^2\acute{d}}{4} + 12\theta\lambda M^4\acute{d}^4\right) \\
& + \left\|\nabla\mathbf{\Gamma}^{(n-\theta)}\right\|^2\left(\theta\lambda + \frac{\delta^2\lambda M^2\acute{d}}{2}\right) + \left\|\nabla\mathbf{\Gamma}^{(n)}\right\|^2\left(3\theta\lambda + 3\frac{\delta^2\theta\lambda\acute{d}M^2}{2}\right) \\
& + \left\|\nabla\mathbf{\Gamma}^{(n+\tilde{\theta})}\right\|^2\left(\tilde{\theta}\lambda + (1-2\theta)\,\lambda + (1-3\theta)\,\frac{\delta^2\lambda M^2\acute{d}}{2} + \frac{(1-2\theta)\,\lambda M^2\acute{d}}{4\epsilon_1} + \frac{2\theta\delta^2\lambda M^2\acute{d}}{2}\right).
\end{aligned}
$$

Multiplying (5.250) by $2\Delta t/\lambda$ and summing from an initial time $t = 0$ to time $t = l$ a telescoping of terms is achieved. An inverse inequality is also applied to the $\|\nabla \boldsymbol{F}\|$ terms on the RHS of the

expression (5.250). Bounds on $\left\|\boldsymbol{F}^{\theta}\right\|^2$ and $\left\|\boldsymbol{F}^{1-\theta}\right\|^2$ (the terms that do not telescope) are stated and proved in the appendix. Thus,

$$
\begin{aligned}
& \left\|\boldsymbol{F}^{l+1}\right\|^2-\left\|\boldsymbol{F}^{0}\right\|^2+\left\|\boldsymbol{F}^{l+1-\theta}\right\|^2-\left\|\boldsymbol{F}^{1-\theta}\right\|^2+\left\|\boldsymbol{F}^{l+\theta}\right\|^2-\left\|\boldsymbol{F}^{\theta}\right\|^2 \\
& +\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\theta\omega\left\|\boldsymbol{F}^{(n+1)}\right\|^2+\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\left(1-2\theta\right)\tilde{\omega}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2+\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\theta\omega\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2 \\
& +\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\left(1-2\theta\right)\left(\delta\lambda-\epsilon_1\delta^2\left(2+3\lambda+2\alpha\right)\right)\left\|\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2 \\
\leq\ & \Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n)}\right\|^2\frac{1}{\lambda}\Bigg(2\theta\lambda+2\theta\omega+3\theta\tilde{\omega}+8\theta\lambda M\acute{d}+8\theta\lambda M^2\acute{d}^2+16\theta\lambda M^4\acute{d}^4+\theta\omega M^2\acute{d}+\frac{3\theta\tilde{\omega}M^2\acute{d}}{2} \\
& \qquad +Ch^{-2}\left(6\lambda\theta\left(2+\delta\acute{d}M^2\right)\right)\Bigg) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n+1)}\right\|^2\frac{1}{\lambda}\Bigg(2+8\lambda+4\alpha+2\theta+2\tilde{\theta}\tilde{\omega}+2\left(1-2\theta\right)\omega+\theta\omega+\frac{\theta\omega M^2\acute{d}}{2}+\lambda\acute{d}M^2+4\lambda\acute{d}M \\
& \qquad +Ch^{-2}\left(2+6\lambda+4\alpha+\delta\lambda\acute{d}M^2+\lambda\acute{d}M^2+2\delta^2\omega\left(2-\theta\right)+4\delta^2\tilde{\omega}+4\delta^2\lambda\right)\Bigg) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n+\theta-1)}\right\|^2\frac{1}{\lambda}\left(\left(1-2\theta\right)\omega 2\left(1+\frac{M^2\acute{d}}{2}\right)\right) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n+\theta)}\right\|^2\frac{1}{\lambda}\Bigg(2+10\lambda+2\theta\omega+2\tilde{\omega}+2\tilde{\theta}\lambda+4\alpha+\theta\omega M^2\acute{d}+2\left(1-2\theta\right)\omega+\tilde{\theta}\omega \\
& \qquad +\frac{\left(1-2\theta\right)\omega M^2\acute{d}}{2}+\frac{\theta\omega M^2\acute{d}}{2}+\omega+\lambda\acute{d}M^2+8\theta\lambda M^4\acute{d}^4+8\theta\lambda M^2\acute{d}^2+\frac{\left(1-2\theta\right)\omega}{2\epsilon_1} \\
& \qquad +Ch^{-2}\left(2+6\lambda+4\alpha+\lambda\acute{d}M^2+\delta\lambda\acute{d}M^2+2\delta^2\left(\omega+2\theta+2\lambda+\left(1-2\theta\right)\omega+2\tilde{\theta}\tilde{\omega}\right)\right)\Bigg) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n-\theta)}\right\|^2\frac{1}{\lambda}\Bigg(\tilde{\omega}+2\theta\lambda+\frac{\tilde{\omega}M^2\acute{d}}{2}+8\tilde{\theta}\lambda M^2\acute{d}^2+8\lambda M^4\acute{d}^4 \\
& \qquad +Ch^{-2}\left(2\lambda+\delta\lambda M^2\acute{d}+2\delta^2\tilde{\theta}\lambda\right)\Bigg) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2\frac{1}{\lambda}\Bigg(2+8\lambda+2\omega+2\theta+2\theta(1-2\omega)+2\left(1-2\theta\right)\lambda+4\alpha+\tilde{\theta}\tilde{\omega}+\left(1-2\theta\right)\tilde{\omega} \\
& \qquad +\frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{2}+\frac{\left(1-2\theta\right)\lambda M^2\acute{d}}{2}+8\tilde{\theta}\lambda\acute{d}M+2\theta\lambda M^2\acute{d}+8\left(1-2\theta\right)\lambda M\acute{d} \\
& \qquad +8\left(1-2\theta\right)\lambda M^4\acute{d}^4+8\tilde{\theta}\lambda M^4\acute{d}^4+32\theta\lambda M^2\acute{d}^2+\frac{\left(1-2\theta\right)\tilde{\omega}}{2\epsilon_1} \\
& \qquad +Ch^{-2}\left(4\lambda+4\theta+6\theta\lambda+8\alpha\theta+\lambda M^2\acute{d}+\theta\lambda M^2\acute{d}+2\theta\delta\lambda M^2\acute{d}+4\delta^2\left(\lambda+2\theta\right)\right)\Bigg) \\
& +\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}H_{\boldsymbol{\Gamma}}+\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}H_g+\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\delta^2H_{upA}+\Delta t\sum_{n=0}^{l}\frac{2}{\lambda}\delta^2H_{upB}. \qquad (5.254)
\end{aligned}
$$

To allow for the application of the discrete Gronwall's Lemma the inequalities stated in (5.255),

(5.256), and (5.257) must be satisfied.

$$\begin{aligned}1 \quad \geq \quad & \Delta t\frac{1}{\lambda}\Bigg(2\theta\lambda + 2\theta\omega + 3\theta\tilde{\omega} + 8\theta\lambda M\acute{d} + 8\theta\lambda M^2\acute{d}^2 + 16\theta\lambda M^4\acute{d}^4 + \theta\omega M^2\acute{d} + \frac{3\theta\tilde{\omega}M^2\acute{d}}{2} \\ & \quad 2 + 8\lambda + 4\alpha + 2\theta + 2\tilde{\theta}\tilde{\omega} + 2\left(1-2\theta\right)\omega + \theta\omega + \frac{\theta\omega M^2\acute{d}}{2} + \lambda\acute{d}M^2 + 4\lambda\acute{d}M \\ & \quad + Ch^{-2}\Bigg(6\lambda\theta\left(2 + \delta\acute{d}M^2\right) + 2 + 6\lambda + 4\alpha + \delta\lambda\acute{d}M^2 \\ & \quad +\lambda\acute{d}M^2 + 2\delta^2\omega\left(2-\theta\right) + 4\delta^2\tilde{\omega} + 4\delta^2\lambda\Bigg)\Bigg)\end{aligned} \tag{5.255}$$

$$\begin{aligned}1 \quad \geq \quad & \Delta t\frac{1}{\lambda}\Bigg(2 + 2\left(1-2\theta\right)\omega\left(1 + \frac{M^2\acute{d}}{2}\right) + 10\lambda + 2\theta\omega + 2\tilde{\omega} + 2\tilde{\theta}\lambda + 4\alpha + \theta\omega M^2\acute{d} + 2\left(1-2\theta\right)\omega \\ & +\tilde{\theta}\omega + \frac{\left(1-2\theta\right)\omega M^2\acute{d}}{2} + \frac{\theta\omega M^2\acute{d}}{2} + \omega + \lambda\acute{d}M^2 + 8\theta\lambda M^4\acute{d}^4 + 8\theta\lambda M^2\acute{d}^2 + \frac{\left(1-2\theta\right)\omega}{2\epsilon_1} \\ & +Ch^{-2}\left(2 + 6\lambda + 4\alpha + \lambda\acute{d}M^2 + \delta\lambda\acute{d}M^2 + 2\delta^2\left(\omega + 2\theta + 2\lambda + \left(1-2\theta\right)\omega + 2\tilde{\theta}\tilde{\omega}\right)\right)\Bigg)\end{aligned} \tag{5.256}$$

$$\begin{aligned}1 \quad \geq \quad & \Delta t\frac{1}{\lambda}\Bigg(2 + \tilde{\omega} + 2\theta\lambda + \frac{\tilde{\omega}M^2\acute{d}}{2} + 8\tilde{\theta}\lambda M^2\acute{d}^2 + 8\lambda M^4\acute{d}^4 \\ & \quad +8\lambda + 2\omega + 2\theta + 2\theta(1-2\omega) + 2\left(1-2\theta\right)\lambda + 4\alpha + \tilde{\theta}\tilde{\omega} + \left(1-2\theta\right)\tilde{\omega} \\ & \quad +\frac{\tilde{\theta}\tilde{\omega}M^2\acute{d}}{2} + \frac{\left(1-2\theta\right)\lambda M^2\acute{d}}{2} + 8\tilde{\theta}\lambda\acute{d}M + 2\theta\lambda M^2\acute{d} + 8\left(1-2\theta\right)\lambda M\acute{d} \\ & \quad +8\left(1-2\theta\right)\lambda M^4\acute{d}^4 + 8\tilde{\theta}\lambda M^4\acute{d}^4 + 32\theta\lambda M^2\acute{d}^2 + \frac{\left(1-2\theta\right)\tilde{\omega}}{2\epsilon_1} \\ & \quad + Ch^{-2}\Bigg(2\lambda + \delta\lambda M^2\acute{d} + 2\delta^2\tilde{\theta}\lambda + 4\lambda + 4\theta + 6\theta\lambda + 8\alpha\theta \\ & \quad +\lambda M^2\acute{d} + \theta\lambda M^2\acute{d} + 2\theta\delta\lambda M^2\acute{d} + 4\delta^2\left(\lambda + 2\theta\right)\Bigg)\Bigg)\end{aligned} \tag{5.257}$$

Summarizing expressions (5.255), (5.256), and (5.257) gives a stability result needed for the discrete Gronwall's Lemma of the form:

$$\Delta t\left(C + Ch^{-2} + \delta C + \delta Ch^{-2} + \delta^2 Ch^{-2}\right) \leq 1. \tag{5.258}$$

This leads to the computationally restrictive stability relationship

$$\Delta t \leq Ch^2. \tag{5.259}$$

At this junction in order to complete the proof of the a priori error estimate we need to consider the bounds on each of the $H_{\mathbf{\Gamma}}, H_g, H_{upA}$, and H_{upB} terms in expression (5.254) using interpolation properties, and some of the lemmas from the appendix.

5.1.8 Bounding $H_{\mathbf{\Gamma}}$

Using interpolation properties of the approximation space, and elements of order m for the stress we approximate the terms in $\Delta t \sum_{n=0}^{l} \frac{2}{\lambda} H_{\mathbf{\Gamma}}$. With C and C_i denoting constants independent of the discretization and upwinding parameters $h, \Delta t$, and δ, exemplary terms are handled as,

$$\begin{aligned}
\sum_{n=0}^{l} \frac{\Delta t}{2} \left\| d_t \left(\mathbf{\Gamma}^{(n+1)} \right) \right\|^2 &= \sum_{n=0}^{l} \frac{\Delta t}{2} \left\| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \frac{\partial \mathbf{\Gamma}}{\partial t} \, dt \right\|^2 && (5.260) \\
&\leq \sum_{n=0}^{l} \frac{1}{2\Delta t} \int_{\Omega} \int_{t_n}^{t_{n+1}} 1^2 \, dt \int_{t_n}^{t_{n+1}} \left(\frac{\partial \mathbf{\Gamma}}{\partial t} \right)^2 dt \, dA \\
&= \sum_{n=0}^{l} \frac{1}{2} \int_{\Omega} \int_{t_n}^{t_{n+1}} \left(\frac{\partial \mathbf{\Gamma}}{\partial t} \right)^2 dt \, dA \\
&\leq C h^{2m+2} \left\| \boldsymbol{\sigma}_t \right\|_{0,m+1}^2 , && (5.261)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{l} \Delta t \, (C_1) \left\| \mathbf{\Gamma}^{(n+\theta)} \right\|^2 &\leq \left(C h^{2m+2} \right) \left(\sum_{n=0}^{l} \Delta t \left\| \boldsymbol{\sigma}^{(n+\theta)} \right\|_{m+1}^2 \right) \\
&\leq C h^{2m+2} \, |||\boldsymbol{\sigma}|||_{0,m+1}^2 , && (5.262)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{l} \Delta t \left(C_2 + \delta^2 C_3 \right) \left\| \nabla \mathbf{\Gamma}^{(n)} \right\|^2 &\leq \left(C h^{2m} + \delta C h^{2m} \right) \left(\sum_{n=0}^{l} \Delta t \left\| \boldsymbol{\sigma}^{(n)} \right\|_{m+1}^2 \right) \\
&\leq C h^{2m} \, |||\boldsymbol{\sigma}|||_{0,m+1}^2 + \delta^2 C h^{2m} \, |||\boldsymbol{\sigma}|||_{0,m+1}^2 , && (5.263)
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta t \sum_{n=0}^{l} \frac{2}{\lambda} H_{\mathbf{\Gamma}} &\leq C \left(h^{2m+2} + h^{2m} + \delta^2 h^{2m} \right) |||\boldsymbol{\sigma}|||_{0,m+1}^2 \\
&\quad + C h^{2m+2} \left\| \boldsymbol{\sigma}_t \right\|_{0,m+1}^2 . && (5.264)
\end{aligned}$$

5.2 Bounding H_g

For each of the terms in H_g a typical term is bounded here. To complete the bounds we use Taylor series expansions. The first bound is shown in detail here, and then the reader is referred to a series

of useful lemmas given in the appendix that aid in bounding the remaining terms. Examine

$$
\begin{aligned}
\sum_{n=0}^{l} \frac{\Delta t}{2} \left\| d_t \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} \right\|^2 &= \sum_{n=0}^{l} \frac{\Delta t}{2} \left\| \frac{1}{\Delta t} \boldsymbol{\sigma}^{(n+1)} - \frac{1}{\Delta t} \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} \right\|^2 \\
&= \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega \left(\frac{1}{\Delta t} \boldsymbol{\sigma}^{(n+1)} - \frac{1}{\Delta t} \boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega \left(\frac{1}{\Delta t} \left(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)} + \frac{\Delta t}{2} \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} + \frac{(\Delta t)^2}{8} \boldsymbol{\sigma}_{tt}^{\left(n+\frac{1}{2}\right)} + \frac{1}{2} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{n+1}-t)^2\, dt \right) \right. \\
&\quad - \frac{1}{\Delta t} \left(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)} - \frac{\Delta t}{2} \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} + \frac{(\Delta t)^2}{8} \boldsymbol{\sigma}_{tt}^{\left(n+\frac{1}{2}\right)} \right. \\
&\quad \left. \left. - \frac{1}{2} \int_{t_n}^{t_{n+\frac{1}{2}}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{(n+1)}-t)^2\, dt \right) - \boldsymbol{\sigma}_t^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega \left(\frac{1}{2\Delta t} \right)^2 \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{n+1}-t)^2\, dt + \int_{t_n}^{t_{n+\frac{1}{2}}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{n+1}-t)^2\, dt \right)^2 dA \\
&\leq \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega 2 \left(\frac{1}{2\Delta t} \right)^2 \left(\left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{(n+1)}-t)^2\, dt \right)^2 \right. \\
&\quad \left. + \left(\int_{t_n}^{t_{n+\frac{1}{2}}} \boldsymbol{\sigma}_{ttt}(\cdot,t)(t_{n+1}-t)^2\, dt \right)^2 \right) dA \\
&\leq \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega 2 \left(\frac{1}{2\Delta t} \right)^2 \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \boldsymbol{\sigma}_{ttt}(\cdot,t)^2\, dt \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (t_{n+1}-t)^4\, dt \right. \\
&\quad \left. + \int_{t_n}^{t_{n+\frac{1}{2}}} \boldsymbol{\sigma}_{ttt}(\cdot,t)^2\, dt \int_{t_n}^{t_{n+\frac{1}{2}}} (t_{n+1}-t)^4\, dt \right) dA \\
&= \sum_{n=0}^{l} \frac{\Delta t}{2} \int_\Omega \frac{\Delta t^3}{5\,(2^6)} \left(\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \boldsymbol{\sigma}_{ttt}(\cdot,t)^2\, dt + \int_{t_n}^{t_{n+\frac{1}{2}}} \boldsymbol{\sigma}_{ttt}(\cdot,t)^2\, dt \right) dA \\
&= \sum_{n=0}^{l} \frac{\Delta t^4}{5\,(2^7)} \int_{t_n}^{t_{n+1}} \left\| \boldsymbol{\sigma}_{ttt}(\cdot,t) \right\|^2\, dt \\
&\leq C(\Delta t)^4 \left\| \boldsymbol{\sigma}_{ttt} \right\|_{0,0}^2 . \qquad (5.265)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{n=0}^{l} \frac{\omega \Delta t}{2\lambda} \left\| \theta \boldsymbol{\sigma}^{(n+\theta)} + \tilde{\theta} \boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)} \right\|^2 &\leq \sum_{n=0}^{l} C(\Delta t)^4 \int_{t_{n+\theta}}^{t_{n+1}} \left\| \boldsymbol{\sigma}_{tt}(\cdot,t) \right\|^2\, dt \qquad (5.266) \\
&\leq C(\Delta t)^4 \left\| \boldsymbol{\sigma}_{tt} \right\|_{0,0}^2 . \qquad (5.267)
\end{aligned}
$$

Next consider the convective term, and use lemma C.2

$$
\begin{aligned}
\sum_{n=0}^{l} \frac{\Delta t}{2} & \left\| \theta \boldsymbol{u}^{(n)} \cdot \nabla \boldsymbol{\sigma}^{(n)} + \tilde{\theta} \boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{\sigma}^{(n+\tilde{\theta})} - \boldsymbol{u}^{(n+\frac{1}{2})} \cdot \nabla \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \\
&\leq \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1-\theta}} \left\| (\boldsymbol{u} \cdot \nabla \boldsymbol{\sigma})_{tt} \right\|^2 \, dt \\
&= \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1-\theta}} \left\| \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma}_{tt} + 2\boldsymbol{u}_t \cdot \nabla \boldsymbol{\sigma}_t + \boldsymbol{u}_{tt} \cdot \nabla \boldsymbol{\sigma} \right\|^2 \, dt \\
&\leq \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1-\theta}} \left(4 \left\| \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma}_{tt} \right\|^2 + 4 \left\| 2\boldsymbol{u}_t \cdot \nabla \boldsymbol{\sigma}_t \right\|^2 + 4 \left\| \boldsymbol{u}_{tt} \cdot \nabla \boldsymbol{\sigma} \right\|^2 \right) dt \\
&\leq \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1-\theta}} \left(4 \left\| \boldsymbol{u} \right\|_\infty^2 \acute{d} \left\| \nabla \boldsymbol{\sigma}_{tt} \right\|^2 + 16 \left\| \boldsymbol{u} \right\|_\infty^2 \acute{d} \left\| \nabla \boldsymbol{\sigma}_t \right\|^2 + 4 \left\| \boldsymbol{u} \right\|_\infty^2 \acute{d} \left\| \nabla \boldsymbol{\sigma} \right\|^2 \right) dt \\
&\leq \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_n}^{t_{n+1-\theta}} \left(4M^2 \acute{d} \left\| \nabla \boldsymbol{\sigma}_{tt} \right\|^2 + 16M^2 \acute{d} \left\| \nabla \boldsymbol{\sigma}_t \right\|^2 + 4M^2 \acute{d} \left\| \nabla \boldsymbol{\sigma} \right\|^2 \right) dt \\
&\leq (\Delta t)^4 C \left(\left\| \boldsymbol{\sigma}_{tt} \right\|_{0,1}^2 + \left\| \boldsymbol{\sigma}_t \right\|_{0,1}^2 + \left\| \boldsymbol{\sigma} \right\|_{0,1}^2 \right).
\end{aligned} \tag{5.268}
$$

Again using lemma (C.2) the $g_a(\cdot, \cdot)$ terms are handled as:

$$
\begin{aligned}
\sum_{n=0}^{l} \frac{\Delta t}{2} & \left\| \theta g_a(\boldsymbol{\sigma}^{(n)}, \nabla \boldsymbol{u}^{(n)}) + \tilde{\theta} g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})}, \nabla \boldsymbol{u}^{(n+\tilde{\theta})}) - g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \nabla \boldsymbol{u}^{(n+\frac{1}{2})}) \right\|^2 \\
&\leq \sum_{n=0}^{l} \frac{(\Delta t)^4}{48} \int_{t_{n+\theta}}^{t_{n+1}} \left\| (g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}))_{tt} \right\|^2 \, dt \\
&\leq \sum_{n=0}^{l} (\Delta t)^4 C_1 \int_{t_{n+\theta}}^{t_{n+1}} \left\| (\boldsymbol{\sigma} \nabla \boldsymbol{u})_{tt} \right\|^2 \, dt \\
&= \sum_{n=0}^{l} (\Delta t)^4 C_1 \int_{t_{n+\theta}}^{t_{n+1}} \left\| \boldsymbol{\sigma} (\nabla \boldsymbol{u})_{tt} + 2\boldsymbol{\sigma}_t (\nabla \boldsymbol{u})_t + \boldsymbol{\sigma}_{tt} \nabla \boldsymbol{u} \right\|^2 \, dt \\
&\leq \sum_{n=0}^{l} (\Delta t)^4 C_2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\left\| \boldsymbol{\sigma} (\nabla \boldsymbol{u})_{tt} \right\|^2 + \left\| \boldsymbol{\sigma}_t (\nabla \boldsymbol{u})_t \right\|^2 + \left\| \boldsymbol{\sigma}_{tt} \nabla \boldsymbol{u} \right\|^2 \right) dt \\
&\leq \sum_{n=0}^{l} (\Delta t)^4 C_2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\left\| (\nabla \boldsymbol{u})_{tt} \right\|_\infty^2 \acute{d}^2 \left\| \boldsymbol{\sigma} \right\|^2 + \left\| (\nabla \boldsymbol{u})_t \right\|_\infty^2 \acute{d}^2 \left\| \boldsymbol{\sigma}_t \right\|^2 + \left\| \nabla \boldsymbol{u} \right\|_\infty^2 \acute{d}^2 \left\| \boldsymbol{\sigma}_{tt} \right\|^2 \right) dt \\
&\leq \sum_{n=0}^{l} (\Delta t)^4 C_2 M^2 \acute{d}^2 \int_{t_{n+\theta}}^{t_{n+1}} \left(\left\| \boldsymbol{\sigma} \right\|^2 + \left\| \boldsymbol{\sigma}_t \right\|^2 + \left\| \boldsymbol{\sigma}_{tt} \right\|^2 \right) dt \\
&\leq (\Delta t)^4 C \left(\left\| \boldsymbol{\sigma}_{tt} \right\|_{0,0}^2 + \left\| \boldsymbol{\sigma}_t \right\|_{0,0}^2 + \left\| \boldsymbol{\sigma} \right\|_{0,0}^2 \right).
\end{aligned} \tag{5.269}
$$

Lastly for the H_g terms we consider

$$\begin{aligned}
\sum_{n=0}^{l} \frac{\alpha\Delta t}{\lambda} \left\| \mathbf{d}(\boldsymbol{u}^{(n+\frac{1}{2})}) - \theta \mathbf{d}(\boldsymbol{u}^{(n)}) - \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\tilde{\theta})}) \right\|^2 & \\
\leq & \sum_{n=0}^{l} \frac{\alpha(\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1-\theta}} \left\| \mathbf{d}(\boldsymbol{u})_{tt} \right\|^2 \, dt \\
\leq & \sum_{n=0}^{l} \frac{\alpha(\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1-\theta}} \left\| (\nabla \boldsymbol{u})_{tt} \right\|^2 \, dt \\
\leq & \sum_{n=0}^{l} \frac{\alpha(\Delta t)^4}{24\lambda} \int_{t_n}^{t_{n+1-\theta}} \left\| (\nabla \boldsymbol{u})_{tt} \right\|_{\infty}^2 \acute{d}^2 \left\| 1 \right\|^2 \, dt \\
\leq & (\Delta t)^4 C C_T, \qquad (5.270)
\end{aligned}$$

where C_T is a constant dependent on the final time T.

In summary we have

$$\begin{aligned}
\Delta t \sum_{n=0}^{l} \frac{2}{\lambda} H_g \leq & \; C(\Delta t)^4 \Big(\left\| \boldsymbol{\sigma}_{ttt} \right\|_{0,0}^2 + \left\| \boldsymbol{\sigma}_{tt} \right\|_{0,1}^2 + \left\| \boldsymbol{\sigma}_{t} \right\|_{0,1}^2 + \left\| \boldsymbol{\sigma} \right\|_{0,1}^2 \\
& + \left\| \boldsymbol{\sigma}_{tt} \right\|_{0,0}^2 + \left\| \boldsymbol{\sigma}_{t} \right\|_{0,0}^2 + \left\| \boldsymbol{\sigma} \right\|_{0,0}^2 + C_T \Big). \qquad (5.271)
\end{aligned}$$

5.3 Bounding H_{upA} and H_{upB}

For each of the H_{upA} terms a characteristic term is treated below. All parts of the H_{Int} terms will match one of the following cases, and is bounded similarly. Thus,

$$\frac{\delta^2 M^2 \acute{d}}{2} \sum_{n=0}^{l} \Delta t \left\| \boldsymbol{\sigma}_t^{(n+\frac{1}{2})} \right\|^2 \leq C\delta^2 \left|\left\| \boldsymbol{\sigma}_t \right\|\right|_{0,0}^2 . \qquad (5.272)$$

Using lemma C.7 we bound

$$\begin{aligned}
\sum_{n=0}^{l} \frac{\delta^2 \omega\theta M^2 \acute{d} \Delta t}{2\lambda} \left\| \boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})} \right\|^2 \leq & \sum_{n=0}^{l} \frac{\delta^2 \omega\theta M^2 \acute{d} (\Delta t)^2}{2\lambda} \int_{t_{n+\theta}}^{t_{n+\frac{1}{2}}} \left\| \boldsymbol{\sigma}_t \right\|^2 \, dt \\
\leq & \; C(\Delta t)^2 \delta^2 \left\| \boldsymbol{\sigma}_t \right\|_{0,0}^2 . \qquad (5.273)
\end{aligned}$$

Using lemma C.8

$$\begin{aligned}
\sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\lambda M^2\acute{d}\Delta t}{2\lambda}\quad & \left\|\boldsymbol{u}^{(n+\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}^{(n+\tilde{\theta})}-\boldsymbol{u}^{(n+\frac{1}{2})}\cdot\nabla\boldsymbol{\sigma}^{(n+\frac{1}{2})}\right\|^2\\
&\leq \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\lambda M^2\acute{d}(\Delta t)^2}{2\lambda}\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma})_t\|^2\ dt\\
&= \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\lambda M^2\acute{d}(\Delta t)^2}{2\lambda}\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|\boldsymbol{u}_t\cdot\nabla\boldsymbol{\sigma}+\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}_t\|^2\ dt\\
&\leq \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\lambda M^2\acute{d}(\Delta t)^2}{\lambda}\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|\boldsymbol{u}_t\cdot\nabla\boldsymbol{\sigma}\|^2+\|\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma}_t\|^2\ dt\\
&\leq C\delta^2(\Delta t)^2\left(\|\boldsymbol{\sigma}\|_{0,1}^2+\|\boldsymbol{\sigma}_t\|_{0,1}^2\right), \qquad (5.274)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{l}\ \frac{\delta^2\tilde{\theta}M^2\acute{d}(\Delta t)}{2} & \left\|g_a(\boldsymbol{\sigma}^{(n+\tilde{\theta})},\nabla\boldsymbol{u}^{(n+\tilde{\theta})})-g_a(\boldsymbol{\sigma}^{(n+\frac{1}{2})},\nabla\boldsymbol{u}^{(n+\frac{1}{2})})\right\|^2\\
&\leq \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}M^2\acute{d}(\Delta t)^2}{2}\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|(g_a(\boldsymbol{\sigma},\nabla\boldsymbol{u}))_t\|^2\ dt\\
&\leq \sum_{n=0}^{l}C_1\delta^2(\Delta t)^2\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|(\boldsymbol{\sigma}\nabla\boldsymbol{u})_t\|^2\ dt\\
&= \sum_{n=0}^{l}C_1\delta^2(\Delta t)^2\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|\boldsymbol{\sigma}_t\nabla\boldsymbol{u}+\boldsymbol{\sigma}\nabla\boldsymbol{u}_t\|^2\ dt\\
&\leq \sum_{n=0}^{l}C_2\delta^2(\Delta t)^2\int_{t_{n+\frac{1}{2}}}^{t_{n+1-\theta}}\|\boldsymbol{\sigma}_t\nabla\boldsymbol{u}\|^2+\|\boldsymbol{\sigma}\nabla\boldsymbol{u}_t\|^2\ dt\\
&\leq C\delta^2(\Delta t)^2\left(\|\boldsymbol{\sigma}_t\|_{0,0}^2+\|\boldsymbol{\sigma}\|_{0,0}^2\right). \qquad (5.275)
\end{aligned}$$

Lastly using lemma C.14

$$\begin{aligned}
\sum_{n=0}^{l}\ \frac{\delta^2\tilde{\theta}\alpha M^2\acute{d}(\Delta t)}{\lambda} & \left\|\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})})-\mathbf{d}(\boldsymbol{u}^{(n-\theta)})\right\|^2\\
&\leq \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\alpha M^2\acute{d}(\Delta t)^2}{\lambda}\int_{t_{n-\theta}}^{t_{n+\theta-\frac{1}{2}}}\|\mathbf{d}(\boldsymbol{u})_t\|^2\ dt\\
&\leq \sum_{n=0}^{l}\frac{\delta^2\tilde{\theta}\alpha M^2\acute{d}(\Delta t)^2}{\lambda}\int_{t_{n-\theta}}^{t_{n+\theta-\frac{1}{2}}}\|(\nabla\boldsymbol{u})_t\|^2\ dt\\
&\leq \delta^2(\Delta t)^2CC_T. \qquad (5.276)
\end{aligned}$$

This gives

$$\sum_{n=0}^{l} \frac{2}{\lambda}\delta^2 \Delta t \left(H_{upA} + H_{upB}\right) \;\; \leq \;\; C\delta^2 (\Delta t)^2 \left(\|\boldsymbol{\sigma}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 \right)$$
$$+ C\delta^2 \left|\left|\left|\boldsymbol{\sigma}_t\right|\right|\right|_{0,0}^2 + \delta^2 (\Delta t)^2 C C_T. \tag{5.277}$$

5.4 Apply the discrete Gronwall's lemma

Using the defined bounds on the H terms, the lemma A.1 and lemma A.2 for bounds on $\left\|\boldsymbol{F}^{(\theta)}\right\|^2$ and $\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2$, and assuming Δt and h satisfy the relationship given in (5.259) the discrete Gronwall's lemma is applied to (5.254) giving

$$\begin{aligned} &\left\|\boldsymbol{F}^{l+1}\right\|^2 + \left\|\boldsymbol{F}^{l+1-\theta}\right\|^2 + \left\|\boldsymbol{F}^{l+\theta}\right\|^2 \\ &\qquad + \Delta t \sum_{n=0}^{l} \frac{2}{\lambda} \left(1 - 2\theta\right) \left(\delta\lambda - \epsilon_1 \delta^2 \left(2 + 3\lambda + 2\alpha\right)\right) \left\|\boldsymbol{u}^{(n+\tilde{\theta})} \cdot \nabla \boldsymbol{F}^{(n+\tilde{\theta})}\right\|^2 \\ &\qquad\qquad \leq G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \end{aligned} \tag{5.278}$$

where

$$\begin{aligned} G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) \;\; := \;\; & C(\Delta t)^4 \Big(\|\boldsymbol{\sigma}_{ttt}\|_{0,0}^2 + \|\boldsymbol{\sigma}_{tt}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,1}^2 \\ & \qquad + \|\boldsymbol{\sigma}_{tt}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + C_T \Big) \\ & + C(\Delta t)^2 \delta^2 \left(\|\boldsymbol{\sigma}\|_{0,1}^2 + \|\boldsymbol{\sigma}_t\|_{0,1}^2 + \|\boldsymbol{\sigma}\|_{0,0}^2 + \|\boldsymbol{\sigma}_t\|_{0,0}^2 + C_T \right) \\ & + C\left(h^{2m+2} + h^{2m} + \delta^2 h^{2m}\right) \left|\left|\left|\boldsymbol{\sigma}\right|\right|\right|_{0,m+1}^2 \\ & + Ch^{2m+2} \|\boldsymbol{\sigma}_t\|_{0,m+1}^2 + \; C\delta^2 \left|\left|\left|\boldsymbol{\sigma}_t\right|\right|\right|_{0,0}^2 . \end{aligned}$$

Using (5.278) it can be seen that

$$\left|\left|\left|\boldsymbol{F}\right|\right|\right|_{\infty,0}^2 \leq G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \tag{5.279}$$

$$\left|\left|\left|\boldsymbol{F}\right|\right|\right|_{0,0}^2 \leq T G_{\boldsymbol{\sigma}}(\Delta t, h, \delta), \tag{5.280}$$

and

$$\left|\left|\left|\boldsymbol{u} \cdot \nabla \boldsymbol{F}\right|\right|\right|_{\tilde{\theta}}^2 \leq \mathcal{C} G_{\boldsymbol{\sigma}}(\Delta t, h, \delta). \tag{5.281}$$

Then

$$\begin{aligned} \left|\left|\left|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\right|\right|\right|_{\infty,0}^2 \;\; & \leq \;\; \left|\left|\left|\boldsymbol{F}\right|\right|\right|_{\infty,0}^2 + \left|\left|\left|\boldsymbol{\Gamma}\right|\right|\right|_{\infty,0}^2 \\ & \leq \;\; G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + Ch^{2m+2} \left|\left|\left|\boldsymbol{\sigma}\right|\right|\right|_{\infty,0}^2 , \end{aligned} \tag{5.282}$$

$$\begin{aligned} \left|\left|\left|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\right|\right|\right|_{0,0}^2 \;\; & \leq \;\; \left|\left|\left|\boldsymbol{F}\right|\right|\right|_{0,0}^2 + \left|\left|\left|\boldsymbol{\Gamma}\right|\right|\right|_{0,0}^2 \\ & \leq \;\; T G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + Ch^{2m+2} \left|\left|\left|\boldsymbol{\sigma}\right|\right|\right|_{0,m+1}^2 , \end{aligned} \tag{5.283}$$

and (see (4.11))

$$
\begin{aligned}
\||\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\||_{\tilde{\theta}}^2 &= \||\boldsymbol{u} \cdot \nabla(\boldsymbol{F} + \boldsymbol{\Gamma})\||_{\tilde{\theta}}^2 \\
&= \||\boldsymbol{u} \cdot \nabla \boldsymbol{F} + \boldsymbol{u} \cdot \nabla \boldsymbol{\Gamma})\||_{\tilde{\theta}}^2 \\
&\le \||\boldsymbol{u} \cdot \nabla \boldsymbol{F}\||_{\tilde{\theta}}^2 + \||\boldsymbol{u} \cdot \nabla \boldsymbol{\Gamma})\||_{\tilde{\theta}}^2 \\
&\le \mathcal{C} G_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + C h^m \||\boldsymbol{\sigma}\||_{0,m+1}^2 \,, \qquad (5.284)
\end{aligned}
$$

establishing the estimates in Theorem 4.1.

■

6 Proof of Theorem 4.2

In this section the proof of Theorem 4.2 is established. The proof is done in a manner similar to that used for Theorem 4.1. For notational simplicity the 'tilde' is dropped from $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}$, the discrete approximations of velocity and pressure using the true stress. We start by examining appropriate linear combinations.

Linear combination 1: Consider the linear combination of

$$
\theta \times (4.3)\Big|_{t=t_n} + (1-2\theta) \times (4.4)\Big|_{t=t_n} + \theta \times (4.7)\Big|_{t=t_n} . \qquad (6.1)
$$

When we multiply the expression resulting from (6.1) be Δt (done in a later step) we have a unit stride of size Δt between t_n and t_{n+1} with

$$
\begin{aligned}
Re\left(\frac{\boldsymbol{u}_h^{(n+1)} - \boldsymbol{u}_h^{(n)}}{\Delta t}, \mathbf{v}\right) + \tilde{\theta}\left(\boldsymbol{\sigma}^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right) + \theta\left(\boldsymbol{\sigma}^{(n+1)}, \mathbf{d}(\mathbf{v})\right) + 2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{u}_h^{(n+1)}), \mathbf{d}(\mathbf{v})\right) \\
+2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right) = \tilde{\theta}\left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) + \theta\left(\mathbf{f}^{(n+1)}, \mathbf{v}\right), \qquad \forall\, \mathbf{v} \in Z_h. \qquad (6.2)
\end{aligned}
$$

$$
\Longrightarrow \quad \mathcal{B}_4^n(\boldsymbol{u}_h, \mathbf{v}) = \tilde{\theta}\left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) + \theta\left(\mathbf{f}^{(n+1)}, \mathbf{v}\right) \qquad (6.3)
$$

Terms in (6.3) are defined by those in (6.2). The true solution of the momentum equation evaluated at the midpoint between t_n and t_{n+1} satisfies:

$$
\begin{aligned}
0 = \left(\mathbf{f}^{(n+\frac{1}{2})}, \mathbf{v}\right) + \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{v}\right) - Re\left(\boldsymbol{u}_t^{(n+\frac{1}{2})}, \mathbf{v}\right) \\
- 2\tilde{\alpha}\left(\mathbf{d}\left(\boldsymbol{u}^{(n+\frac{1}{2})}\right), \mathbf{d}(\mathbf{v})\right) - \left(\boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{v})\right), \ \ \forall\, \mathbf{v} \in Z_h. \qquad (6.4)
\end{aligned}
$$

Adding $\mathcal{B}_4^n(\boldsymbol{u}, \mathbf{v})$ to both sides (6.4) yields

$$
\mathcal{B}_4^n(\boldsymbol{u}, \mathbf{v}) = \left(\mathbf{f}^{(n+\frac{1}{2})}, \mathbf{v}\right) + \left(p^{(n+\frac{1}{2})}, \nabla \cdot \mathbf{v}\right) + \mathcal{I}_4^n(\mathbf{v}), \ \ \forall\, \mathbf{v} \in Z_h, \qquad (6.5)
$$

where the term

$$
\begin{aligned}
\mathcal{I}_4^n(\mathbf{v}) := Re\left(d_t \boldsymbol{u}^{(n+1)} - \boldsymbol{u}_t^{(n+\frac{1}{2})}, \mathbf{v}\right) + \left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{(n+\frac{1}{2})}, \mathbf{d}(\mathbf{v})\right) \\
+ 2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) - \mathbf{d}\left(\boldsymbol{u}^{(n+\frac{1}{2})}\right), \mathbf{d}(\mathbf{v})\right), \ \ \forall\, \mathbf{v} \in Z_h. \qquad (6.6)
\end{aligned}
$$

Subtraction of (6.2) from (6.5) gives the error equation:

$$\mathcal{B}_4^n(e_{\boldsymbol{u}}, \mathbf{v}) = \left(\mathbf{f}^{\left(n+\frac{1}{2}\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n+1)}, \mathbf{v}\right) + \left(p^{\left(n+\frac{1}{2}\right)}, \nabla\cdot\mathbf{v}\right) + \mathcal{I}_4^n(\mathbf{v}), \quad \forall\, \mathbf{v} \in Z_h. \tag{6.7}$$

This procedure is now repeated, setting up two more error expressions.

Linear combination 2: Using the linear combination of

$$\theta \times (4.3)\Big|_{t=t_n} + (1-2\theta)\times(4.4)\Big|_{t=t_n} + \theta\times(4.7)\Big|_{t=t_{n-1}}, \tag{6.8}$$

will provide a unit stride between $t_{n-\theta}$ and $t_{n+1-\theta}$ when we multiply the resulting expression by Δt in a later step.

$$\begin{aligned} Re\left(\frac{\boldsymbol{u}_h^{(n+\tilde{\theta})} - \boldsymbol{u}_h^{(n-\theta)}}{\Delta t}, \mathbf{v}\right) + \tilde{\theta}\left(\boldsymbol{\sigma}^{(n+\theta)}, \mathbf{d}(\mathbf{v})\right) + \theta\left(\boldsymbol{\sigma}^{(n)}, \mathbf{d}(\mathbf{v})\right) + \tilde{\theta}2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}), \mathbf{d}(\mathbf{v})\right) \\ + \theta 2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{u}_h^{(n)}), \mathbf{d}(\mathbf{v})\right) = \tilde{\theta}\left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) + \theta\left(\mathbf{f}^{(n)}, \mathbf{v}\right), \qquad \forall\, \mathbf{v} \in Z_h. \end{aligned} \tag{6.9}$$

$$\Longrightarrow \quad \mathcal{B}_5^n(\boldsymbol{u}_h, \mathbf{v}) = \tilde{\theta}\left(\mathbf{f}^{(n+\theta)}, \mathbf{v}\right) + \theta\left(\mathbf{f}^{(n)}, \mathbf{v}\right). \tag{6.10}$$

Again the terms in (6.10) are defined by those in (6.9). The true solution of the momentum equation evaluated at the midpoint between $t_{n-\theta}$ and $t_{n+1-\theta}$ satisfies:

$$\begin{aligned} 0 = \left(\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{v}\right) + \left(p^{\left(n+\frac{1}{2}-\theta\right)}, \nabla\cdot\mathbf{v}\right) - Re\left(\boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{v}\right) \\ - 2\tilde{\alpha}\left(\mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right), \mathbf{d}(\mathbf{v})\right) - \left(\boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{d}(\mathbf{v})\right), \quad \forall\, \mathbf{v} \in Z_h. \end{aligned} \tag{6.11}$$

$\mathcal{B}_5^n(\boldsymbol{u}, \mathbf{v})$ is now added to both sides (6.11) giving

$$\mathcal{B}_5^n(\boldsymbol{u}, \mathbf{v}) = \left(\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{v}\right) + \left(p^{\left(n+\frac{1}{2}-\theta\right)}, \nabla\cdot\mathbf{v}\right) + \mathcal{I}_5^n(\mathbf{v}). \tag{6.12}$$

Here we define

$$\begin{aligned} \mathcal{I}_5^n(\mathbf{v}) := Re\left(d_t\boldsymbol{u}^{(n+\tilde{\theta})} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{v}\right) + \left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{d}(\mathbf{v})\right) \\ + 2\tilde{\alpha}\left(\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right), \mathbf{d}(\mathbf{v})\right), \quad \forall\, \mathbf{v} \in Z_h. \end{aligned} \tag{6.13}$$

Lastly the error expression is obtained by subtracting (6.9) from (6.12) giving

$$\mathcal{B}_5^n(e_{\boldsymbol{u}}, \mathbf{v}) = \left(\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n)}, \mathbf{v}\right) + \left(p^{\left(n+\frac{1}{2}-\theta\right)}, \nabla\cdot\mathbf{v}\right) + \mathcal{I}_5^n(\mathbf{v}), \quad \forall\, \mathbf{v} \in Z_h. \tag{6.14}$$

Linear combination 3: Multiplying the linear combination of

$$\theta \times (4.3)\Big|_{t=t_n} + (1-2\theta)\times(4.4)\Big|_{t=t_{n-1}} + \theta\times(4.7)\Big|_{t=t_{n-1}}, \tag{6.15}$$

in a later step by Δt provides a unit stride between $t_{n+\theta-1}$ and $t_{n+\theta}$.

$$
\begin{aligned}
&Re\left(\frac{\boldsymbol{u}_h^{(n+\theta)}-\boldsymbol{u}_h^{(n+\theta-1)}}{\Delta t},\mathbf{v}\right)+2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta)}),\mathbf{d}(\mathbf{v})\right)+2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{u}_h^{(n)}),\mathbf{d}(\mathbf{v})\right)\\
&\quad+2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(\boldsymbol{u}_h^{(n+\theta-1)}),\mathbf{d}(\mathbf{v})\right)+\theta\left(\boldsymbol{\sigma}^{(n+\theta)},\mathbf{d}(\mathbf{v})\right)+(1-2\theta)\left(\boldsymbol{\sigma}^{(n+\theta-1)},\mathbf{d}(\mathbf{v})\right)\\
&+\theta\left(\boldsymbol{\sigma}^{(n)},\mathbf{d}(\mathbf{v})\right)=\theta\left(\mathbf{f}^{(n+\theta)},\mathbf{v}\right)+(1-2\theta)\left(\mathbf{f}^{(n+\theta-1)},\mathbf{v}\right)+\theta\left(\mathbf{f}^{(n)},\mathbf{v}\right),\qquad\forall\,\mathbf{v}\in Z_h.
\end{aligned}
\tag{6.16}
$$

$$
\implies\quad \mathcal{B}_6^n(\boldsymbol{u}_h,\mathbf{v})=\theta\left(\mathbf{f}^{(n+\theta)},\mathbf{v}\right)+(1-2\theta)\left(\mathbf{f}^{(n+\theta-1)},\mathbf{v}\right)+\theta\left(\mathbf{f}^{(n)},\mathbf{v}\right).
\tag{6.17}
$$

The terms in (6.17) are defined by those in (6.16). The true solution of the momentum equation evaluated at the midpoint between $t_{n+\theta-1}$ and $t_{n+\theta}$ satisfies:

$$
\begin{aligned}
0=&\left(\mathbf{f}^{(\mathbf{n}+\theta-\frac{1}{2})},\mathbf{v}\right)+\left(p^{(n+\theta-\frac{1}{2})},\nabla\cdot\mathbf{v}\right)-Re\left(\boldsymbol{u}_t^{(n+\theta-\frac{1}{2})},\mathbf{v}\right)\\
&-2\tilde{\alpha}\left(\mathbf{d}\left(\boldsymbol{u}^{(n+\theta-\frac{1}{2})}\right),\mathbf{d}(\mathbf{v})\right)-\left(\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})},\mathbf{d}(\mathbf{v})\right),\ \ \forall\,\mathbf{v}\in Z_h.
\end{aligned}
\tag{6.18}
$$

Adding $\mathcal{B}_6^n(\boldsymbol{u},\mathbf{v})$ to both sides of (6.18) we obtain:

$$
\mathcal{B}_6^n(\boldsymbol{u},\mathbf{v})=\left(\mathbf{f}^{(\mathbf{n}+\theta-\frac{1}{2})},\mathbf{v}\right)+\left(p^{(n+\theta-\frac{1}{2})},\nabla\cdot\mathbf{v}\right)+\mathcal{I}_6^n(\mathbf{v}),\quad\forall\mathbf{v}\in Z_h,
\tag{6.19}
$$

with

$$
\begin{aligned}
\mathcal{I}_6^n(\mathbf{v}):=&\,Re\left(d_t\boldsymbol{u}^{(n+\theta)}-\boldsymbol{u}_t^{(n+\theta-\frac{1}{2})},\mathbf{v}\right)+\left(\theta\boldsymbol{\sigma}^{(n+\theta)}+(1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)}+\theta\boldsymbol{\sigma}^{(n)}-\boldsymbol{\sigma}^{(n+\theta-\frac{1}{2})},\mathbf{d}(\mathbf{v})\right)\\
&+2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+\theta)})+\theta\mathbf{d}(\boldsymbol{u}^{(n)})+(1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\theta-1)})-\mathbf{d}(\boldsymbol{u}^{(n+\theta-\frac{1}{2})}),\mathbf{d}(\mathbf{v})\right),\forall\,\mathbf{v}\in Z_h.
\end{aligned}
\tag{6.20}
$$

Subtracting (6.16) from (6.19) gives the final error expression for the conservation equations.

$$
\begin{aligned}
\mathcal{B}_6^n(e_{\boldsymbol{u}},\mathbf{v})=&\left(\mathbf{f}^{(\mathbf{n}+\theta-\frac{1}{2})}-\theta\mathbf{f}^{(n+\theta)}-(1-2\theta)\,\mathbf{f}^{(n+\theta-1)}-\theta\mathbf{f}^{(n)},\mathbf{v}\right)\\
&+\left(p^{(n+\frac{1}{2}-\theta)},\nabla\cdot\mathbf{v}\right)+\mathcal{I}_6^n(\mathbf{v}),\quad\forall\mathbf{v}\in Z_h.
\end{aligned}
\tag{6.21}
$$

In order to complete the analysis bounds are now found on (6.7),(6.14), and (6.21), with the value of the test function $\mathbf{v}$ set as $\boldsymbol{E}^{(n+1)}$,$\boldsymbol{E}^{(n+\tilde{\theta})}$, and $\boldsymbol{E}^{(n+\theta)}$ respectively.

6.1 Bounding $\mathcal{B}_4^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+1)})$, $\mathcal{B}_5^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+\tilde{\theta})})$, and $\mathcal{B}_6^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+\theta)})$

In order to complete the analysis for the conservation equation we need to consider bounds on (6.7),(6.14), and (6.21). In order to consider the bounds on the terms in $\mathcal{B}_4^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+1)})$, $\mathcal{B}_5^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+\tilde{\theta})})$, and $\mathcal{B}_6^n(e_{\boldsymbol{u}},\boldsymbol{E}^{(n+\theta)})$ we use the following estimates:

$$
Re\left(d_te_{\boldsymbol{u}}^{(n+1)},\boldsymbol{E}^{(n+1)}\right)=Re\left(d_t\boldsymbol{E}^{(n+1)},\boldsymbol{E}^{(n+1)}\right)+Re\left(d_t\boldsymbol{\Lambda}^{(n+1)},\boldsymbol{E}^{(n+1)}\right),
\tag{6.22}
$$

with bounds

$$
Re\left(d_t\boldsymbol{E}^{(n+1)},\boldsymbol{E}^{(n+1)}\right)\geq\frac{Re}{2\Delta t}\left(\left\|\boldsymbol{E}^{(n+1)}\right\|^2-\left\|\boldsymbol{E}^{(n)}\right\|^2\right),
\tag{6.23}
$$

and

$$Re\left(d_t\mathbf{\Lambda}^{(n+1)},\boldsymbol{E}^{(n+1)}\right)\leq\frac{Re}{4}\left\|d_t\mathbf{\Lambda}^{(n+1)}\right\|^2+Re\left\|E^{(n+1)}\right\|^2. \tag{6.24}$$

Similarly,

$$Re\left(d_te_{\boldsymbol{u}}^{(n+\tilde{\theta})},\boldsymbol{E}^{(n+\tilde{\theta})}\right)=Re\left(d_t\boldsymbol{E}^{(n+\tilde{\theta})},\boldsymbol{E}^{(n+\tilde{\theta})}\right)+Re\left(d_t\mathbf{\Lambda}^{(n+\tilde{\theta})},\boldsymbol{E}^{(n+\tilde{\theta})}\right), \tag{6.25}$$

and is bounded as

$$Re\left(d_t\boldsymbol{E}^{(n+\tilde{\theta})},\boldsymbol{E}^{(n+\tilde{\theta})}\right)\geq\frac{Re}{2\Delta t}\left(\left\|\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2-\left\|\boldsymbol{E}^{(n-\theta)}\right\|^2\right), \tag{6.26}$$

and

$$Re\left(d_t\mathbf{\Lambda}^{(n+\tilde{\theta})},\boldsymbol{E}^{(n+\tilde{\theta})}\right)\leq\frac{Re}{4}\left\|d_t\mathbf{\Lambda}^{(n+\tilde{\theta})}\right\|^2+Re\left\|\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2. \tag{6.27}$$

We bound

$$Re\left(d_te_{\boldsymbol{u}}^{(n+\theta)},\boldsymbol{E}^{(n+\theta)}\right)=Re\left(d_t\boldsymbol{E}^{(n+\theta)},\boldsymbol{E}^{(n+\theta)}\right)+Re\left(d_t\mathbf{\Lambda}^{(n+\theta)},\boldsymbol{E}^{(n+\theta)}\right), \tag{6.28}$$

as

$$Re\left(d_t\boldsymbol{E}^{(n+\theta)},\boldsymbol{E}^{(n+\theta)}\right)\geq\frac{Re}{2\Delta t}\left(\left\|\boldsymbol{E}^{(n+\theta)}\right\|^2-\left\|\boldsymbol{E}^{(n+\theta-1)}\right\|^2\right), \tag{6.29}$$

and

$$Re\left(d_t\mathbf{\Lambda}^{(n+\theta)},\boldsymbol{E}^{(n+\theta)}\right)\leq\frac{Re}{4}\left\|d_t\mathbf{\Lambda}^{(n+\theta)}\right\|^2+Re\left\|\boldsymbol{E}^{(n+\theta)}\right\|^2. \tag{6.30}$$

For the deformation tensor terms we have,

$$2\tilde{\alpha}\theta\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n+1)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)=2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n+1)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)+2\tilde{\alpha}\theta\left(\mathbf{d}(\mathbf{\Lambda}^{(n+1)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right), \tag{6.31}$$

with bounds

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n+1)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)=2\tilde{\alpha}\theta\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2, \tag{6.32}$$

and

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\mathbf{\Lambda}^{(n+1)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)\leq\frac{2\tilde{\alpha}\theta}{4\epsilon_2}\left\|\nabla\mathbf{\Lambda}^{(n+1)}\right\|^2+2\tilde{\alpha}\theta\epsilon_2\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2. \tag{6.33}$$

The term

$$2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)=2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)+2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right), \tag{6.34}$$

and is bounded such that

$$2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)\leq\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\left\|\nabla\boldsymbol{E}^{(n+\theta)}\right\|^2+2\tilde{\alpha}\tilde{\theta}\epsilon_2\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2, \tag{6.35}$$

and

$$2\tilde{\alpha}\tilde{\theta}\left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+1)})\right)\leq\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\left\|\nabla\mathbf{\Lambda}^{(n+\theta)}\right\|^2+2\tilde{\alpha}\tilde{\theta}\epsilon_2\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2. \tag{6.36}$$

We have

$$\begin{aligned}&(1-\theta)2\tilde{\alpha}\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)\\&\qquad=(1-\theta)2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)+(1-\theta)2\tilde{\alpha}\left(\mathbf{d}(\mathbf{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right),\end{aligned} \tag{6.37}$$

with bounds

$$(1-\theta)2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)\leq\frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3}\left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2+(1-\theta)2\tilde{\alpha}\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2, \quad (6.38)$$

and

$$(1-\theta)2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)\leq\frac{(1-\theta)2\tilde{\alpha}}{2}\left\|\nabla\boldsymbol{\Lambda}^{(n+\theta)}\right\|^2+\frac{(1-\theta)2\tilde{\alpha}}{2}\left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2. \quad (6.39)$$

Next

$$\theta2\tilde{\alpha}\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)=\theta2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{E}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)+\theta2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right), \quad (6.40)$$

with

$$\theta2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{E}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)\leq\theta\tilde{\alpha}\left\|\nabla\boldsymbol{E}^{(n)}\right\|^2+\theta\tilde{\alpha}\left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2, \quad (6.41)$$

and

$$\theta2\tilde{\alpha}\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right)\leq\theta\tilde{\alpha}\left\|\nabla\boldsymbol{\Lambda}^{(n)}\right\|^2+\theta\tilde{\alpha}\left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2. \quad (6.42)$$

Splitting

$$2\tilde{\alpha}\theta\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)=2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)+2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right), \quad (6.43)$$

with bounds

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)=2\tilde{\alpha}\theta\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2, \quad (6.44)$$

and

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n+\theta)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\leq\frac{\tilde{\alpha}\theta}{2\epsilon_3}\left\|\nabla\boldsymbol{\Lambda}^{(n+\theta)}\right\|^2+2\tilde{\alpha}\theta\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2. \quad (6.45)$$

Then

$$2\tilde{\alpha}\theta\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)=2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)+2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right), \quad (6.46)$$

with each terms handled as:

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{E}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\leq\frac{\tilde{\alpha}\theta}{2\epsilon_3}\left\|\nabla\boldsymbol{E}^{(n)}\right\|^2+2\tilde{\alpha}\theta\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2, \quad (6.47)$$

$$2\tilde{\alpha}\theta\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\leq\frac{2\tilde{\alpha}\theta}{4\epsilon_3}\left\|\nabla\boldsymbol{\Lambda}^{(n)}\right\|^2+2\tilde{\alpha}\theta\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2. \quad (6.48)$$

Lastly

$$\begin{aligned}&2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(e_{\boldsymbol{u}}^{(n+\theta-1)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\\&\quad=2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta-1)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)+2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n+\theta-1)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right), \quad (6.49)\end{aligned}$$

where

$$\begin{aligned}&2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(\boldsymbol{E}^{(n+\theta-1)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\\&\qquad\leq\frac{\tilde{\alpha}\left(1-2\theta\right)}{2\epsilon_3}\left\|\nabla\boldsymbol{E}^{(n+\theta-1)}\right\|^2+2\tilde{\alpha}\left(1-2\theta\right)\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2, \quad (6.50)\end{aligned}$$

and

$$\begin{aligned}&2\tilde{\alpha}\left(1-2\theta\right)\left(\mathbf{d}(\boldsymbol{\Lambda}^{(n+\theta-1)}),\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right)\\&\qquad\leq\frac{\tilde{\alpha}\left(1-2\theta\right)}{2\epsilon_3}\left\|\nabla\boldsymbol{\Lambda}^{(n+\theta-1)}\right\|+2\tilde{\alpha}\left(1-2\theta\right)\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2. \quad (6.51)\end{aligned}$$

6.2 Bounding $\mathcal{I}_4^n(\boldsymbol{E}^{(n+1)})$

Bounds on the interpolation error are now considered for $\mathcal{I}_4^n(\boldsymbol{E}^{(n+1)})$ (see (6.6)). Thus,

$$\mathcal{I}_4^n(\boldsymbol{E}^{(n+1)}) := Re\left(d_t\boldsymbol{u}^{(n+1)} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}\right)}, \boldsymbol{E}^{(n+1)}\right) + \left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \mathbf{d}(\boldsymbol{E}^{(n+1)})\right) \\ + 2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\right), \mathbf{d}(\boldsymbol{E}^{(n+1)})\right). \quad (6.52)$$

We bound each of these terms as

$$Re\left(d_t\boldsymbol{u}^{(n+1)} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}\right)}, \boldsymbol{E}^{(n+1)}\right) \le \frac{Re}{4}\left\|d_t\boldsymbol{u}^{(n+1)} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}\right)}\right\|^2 + Re\left\|\boldsymbol{E}^{(n+1)}\right\|^2, \quad (6.53)$$

$$\left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}, \mathbf{d}(\boldsymbol{E}^{(n+1)})\right) \\ \le \frac{1}{4\epsilon_2}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 + \epsilon_2\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2, \quad (6.54)$$

and

$$2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\right), \mathbf{d}(\boldsymbol{E}^{(n+1)})\right) \\ \le \frac{\tilde{\alpha}}{2\epsilon_2}\left\|\theta\mathbf{d}(\boldsymbol{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\right)\right\|^2 + 2\tilde{\alpha}\epsilon_2\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2. \quad (6.55)$$

6.3 Bounding $\mathcal{I}_5^n(\boldsymbol{E}^{\left(n+\tilde{\theta}\right)})$

Similar to the previous section we now bound the $\mathcal{I}_5^n$ term (see (6.13)). Here

$$\mathcal{I}_5^n(\boldsymbol{E}^{(n+\tilde{\theta})}) := Re\left(d_t\boldsymbol{u}^{(n+\tilde{\theta})} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{E}^{(n+\tilde{\theta})}\right) + \left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right) \\ + 2\tilde{\alpha}\left(\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right), \mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right). \quad (6.56)$$

Then

$$Re\left(d_t\boldsymbol{u}_h^{(n+\tilde{\theta})} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}, \boldsymbol{E}^{(n+\tilde{\theta})}\right) \le \frac{Re}{4}\left\|d_t\boldsymbol{u}^{(n+\tilde{\theta})} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + Re\left\|\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2, \quad (6.57)$$

$$\left(\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}, \mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right) \\ \le \frac{1}{4}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2, \quad (6.58)$$

and

$$2\tilde{\alpha}\left(\tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right), \mathbf{d}(\boldsymbol{E}^{(n+\tilde{\theta})})\right) \\ \le \frac{2\tilde{\alpha}}{4}\left\|(1-\theta)\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right)\right\|^2 + 2\tilde{\alpha}\left\|\nabla\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2. \quad (6.59)$$

6.4 Bounding $\mathcal{I}_6^n(\boldsymbol{E}^{(n+\theta)})$

Here we examin the bounds given for the $\mathcal{I}_6^n$ term (see (6.20)) with the choice of $\mathbf{v} = \boldsymbol{E}^{(n+\theta)}$.

$$\begin{aligned}\mathcal{I}_6^n(\boldsymbol{E}^{(n+\theta)}) := Re\left(d_t\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}_t^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{E}^{(n+\theta)}\right) \\ + \left(\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right) \\ + 2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) + (1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\theta-1)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right).\end{aligned} \tag{6.60}$$

Each piece is bounded as

$$Re\left(d_t\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}_t^{\left(n+\theta-\frac{1}{2}\right)}, \boldsymbol{E}^{(n+\theta)}\right) \le \frac{Re}{4}\left\|d_t\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + Re\left\|\boldsymbol{E}^{(n+\theta)}\right\|^2, \tag{6.61}$$

$$\begin{aligned}&\left(\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}, \mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right) \\ &\qquad \le \frac{1}{4\epsilon_3}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2,\end{aligned} \tag{6.62}$$

and

$$\begin{aligned}&2\tilde{\alpha}\left(\theta\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) + (1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\theta-1)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)}), \mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right) \\ &\le \frac{\tilde{\alpha}}{2\epsilon_3}\left\|\theta\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) + (1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\theta-1)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2 + 2\tilde{\alpha}\epsilon_3\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2.\end{aligned} \tag{6.63}$$

6.5 Bounding the f and p terms

Bounds for the terms containing the forcing function $\mathbf{f}$ are

$$\left(\mathbf{f}^{\left(n+\frac{1}{2}\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n+1)}, \boldsymbol{E}^{(n+1)}\right) \le \frac{1}{4}\left\|\mathbf{f}^{\left(n+\frac{1}{2}\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n+1)}\right\|^2 + \left\|\boldsymbol{E}^{(n+1)}\right\|^2, \tag{6.64}$$

$$\left(\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n)}, \boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right) \le \frac{1}{4}\left\|\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)} - \tilde{\theta}\mathbf{f}^{(n+\theta)} - \theta\mathbf{f}^{(n)}\right\|^2 + \left\|\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{6.65}$$

and

$$\begin{aligned}&\left(\mathbf{f}^{\left(n+\theta-\frac{1}{2}\right)} - \theta\mathbf{f}^{(n+\theta)} - (1-2\theta)\,\mathbf{f}^{(n+\theta-1)} - \theta\mathbf{f}^{(n)}, \boldsymbol{E}^{(n+\theta)}\right) \\ &\qquad \le \frac{1}{4}\left\|\mathbf{f}^{\left(n+\theta-\frac{1}{2}\right)} - \theta\mathbf{f}^{(n+\theta)} - (1-2\theta)\,\mathbf{f}^{(n+\theta-1)} - \theta\mathbf{f}^{(n)}\right\|^2 + \left\|\boldsymbol{E}^{(n+\theta)}\right\|^2.\end{aligned} \tag{6.66}$$

The pressure terms are bounded using $p^{(\cdot)} \in Z_h$

$$\begin{aligned}\left(p^{\left(n+\frac{1}{2}\right)}, \nabla\cdot\boldsymbol{E}^{(n+1)}\right) &= \left(p^{\left(n+\frac{1}{2}\right)} - \mathcal{P}^{\left(n+\frac{1}{2}\right)}, \nabla\cdot\boldsymbol{E}^{(n+1)}\right) \\ &\le \left\|p^{\left(n+\frac{1}{2}\right)} - \mathcal{P}^{\left(n+\frac{1}{2}\right)}\right\| \acute{d}^{\frac{1}{2}}\left\|\nabla\boldsymbol{E}^{(n+1)}\right\|^2 \\ &\le \frac{\acute{d}^{\frac{1}{2}}}{4}\left\|p^{\left(n+\frac{1}{2}\right)} - \mathcal{P}^{\left(n+\frac{1}{2}\right)}\right\|^2 + \left\|\nabla\boldsymbol{E}^{(n+1)}\right\|^2,\end{aligned} \tag{6.67}$$

$$\left(p^{\left(n+\frac{1}{2}-\theta\right)}, \nabla\cdot\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right) \leq \frac{\acute{d}^{\frac{1}{2}}}{4}\left\|p^{\left(n+\frac{1}{2}-\theta\right)} - \mathcal{P}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \left\|\nabla\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right\|^2, \tag{6.68}$$

and

$$\left(p^{\left(n+\theta-\frac{1}{2}\right)}, \nabla\cdot\boldsymbol{E}^{\left(n+\theta\right)}\right) \leq \frac{\acute{d}^{\frac{1}{2}}}{4}\left\|p^{\left(n+\theta-\frac{1}{2}\right)} - \mathcal{P}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 + \left\|\nabla\boldsymbol{E}^{\left(n+\theta\right)}\right\|^2. \tag{6.69}$$

6.6 Bring all the parts together for the conservation equations

Using the estimates given in (6.22) - (6.69) a single error expression is given by

$$\begin{aligned}
&\frac{Re}{2\Delta t}\left(\left\|\boldsymbol{E}^{(n+1)}\right\|^2 - \left\|\boldsymbol{E}^{(n)}\right\|^2 + \left\|\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right\|^2 - \left\|\boldsymbol{E}^{(n-\theta)}\right\|^2 + \left\|\boldsymbol{E}^{(n+\theta)}\right\|^2 - \left\|\boldsymbol{E}^{(n+\theta-1)}\right\|^2\right)\\
&+\left(2\tilde{\alpha}\theta - \epsilon_2\left(\tilde{\alpha}\theta + 2\tilde{\alpha} + 4\tilde{\alpha}\tilde{\theta} + 1\right)\right)\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2\\
&+\left(2\tilde{\alpha}\theta - \epsilon_3\left(2\tilde{\alpha}(1-\theta) + 6\tilde{\alpha}\theta + 4\tilde{\alpha}\left(1-2\theta\right) + 2\tilde{\alpha} + 1\right)\right)\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2\\
\leq\;& (2Re+1)\left\|\boldsymbol{E}^{(n+1)}\right\|^2 + (2Re+1)\left\|\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right\|^2 + (2Re+1)\left\|\boldsymbol{E}^{(n+\theta)}\right\|^2\\
&+\left(\theta\tilde{\alpha} + \frac{\tilde{\alpha}\theta}{2\epsilon_3}\right)\left\|\nabla\boldsymbol{E}^{(n)}\right\|^2 + \left(1 + \frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\right)\left\|\nabla\boldsymbol{E}^{(n+\theta)}\right\|^2 + \frac{\tilde{\alpha}\left(1-2\theta\right)}{2\epsilon_3}\left\|\nabla\boldsymbol{E}^{(n+\theta-1)}\right\|^2\\
&+\left(2 + (1+\theta)2\tilde{\alpha} + \frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3} + \frac{(1-\theta)2\tilde{\alpha}}{2}\right)\left\|\nabla\boldsymbol{E}^{\left(n+\tilde{\theta}\right)}\right\|^2 + \left\|\nabla\boldsymbol{E}^{(n+1)}\right\|^2\\
&+H_{\boldsymbol{\Lambda}} + H_{\mathcal{I}^n} + H_{\mathbf{f}} + H_p,
\end{aligned} \tag{6.70}$$

where

$$\begin{aligned}
H_{\boldsymbol{\Lambda}} \;:=\;& \frac{Re}{4}\left(\left\|d_t\boldsymbol{\Lambda}^{(n+1)}\right\|^2 + \left\|d_t\boldsymbol{\Lambda}^{\left(n+\tilde{\theta}\right)}\right\|^2 + \left\|d_t\boldsymbol{\Lambda}^{(n+\theta)}\right\|^2\right)\\
&+\frac{2\tilde{\alpha}\theta}{4\epsilon_2}\left\|\nabla\boldsymbol{\Lambda}^{(n+1)}\right\|^2 + \left(\theta\tilde{\alpha} + \frac{2\tilde{\alpha}\theta}{4\epsilon_3}\right)\left\|\nabla\boldsymbol{\Lambda}^{(n)}\right\|^2 + \frac{\tilde{\alpha}\left(1-2\theta\right)}{2\epsilon_3}\left\|\nabla\boldsymbol{\Lambda}^{(n+\theta-1)}\right\|\\
&+\left(\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2} + \frac{(1-\theta)2\tilde{\alpha}}{2} + \frac{\tilde{\alpha}\theta}{2\epsilon_3}\right)\left\|\nabla\boldsymbol{\Lambda}^{(n+\theta)}\right\|^2,
\end{aligned} \tag{6.71}$$

$$\begin{aligned}
H_{\mathcal{I}^n} \;:=\;& \frac{Re}{4}\left(\left\|d_t\boldsymbol{u}^{(n+1)} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}\right)}\right\|^2 + \left\|d_t\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}_t^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 + \left\|d_t\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}_t^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2\right)\\
&+\frac{1}{4\epsilon_2}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}\right)}\right\|^2 + \frac{1}{4}\left\|\tilde{\theta}\boldsymbol{\sigma}^{(n+\theta)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2\\
&+\frac{1}{4\epsilon_3}\left\|\theta\boldsymbol{\sigma}^{(n+\theta)} + (1-2\theta)\,\boldsymbol{\sigma}^{(n+\theta-1)} + \theta\boldsymbol{\sigma}^{(n)} - \boldsymbol{\sigma}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2\\
&+\frac{\tilde{\alpha}}{2\epsilon_2}\left\|\theta\mathbf{d}(\boldsymbol{u}^{(n+1)}) + \tilde{\theta}\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}\right)}\right)\right\|^2\\
&+\frac{\tilde{\alpha}}{2}\left\|(1-\theta)\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) - \mathbf{d}\left(\boldsymbol{u}^{\left(n+\frac{1}{2}-\theta\right)}\right)\right\|^2\\
&+\frac{\tilde{\alpha}}{2\epsilon_3}\left\|\theta\mathbf{d}(\boldsymbol{u}^{(n+\theta)}) + \theta\mathbf{d}(\boldsymbol{u}^{(n)}) + (1-2\theta)\,\mathbf{d}(\boldsymbol{u}^{(n+\theta-1)}) - \mathbf{d}(\boldsymbol{u}^{\left(n+\theta-\frac{1}{2}\right)})\right\|^2,
\end{aligned} \tag{6.72}$$

$$
\begin{aligned}
H_{\mathbf{f}} \;:=\; & \frac{1}{4}\left\|\mathbf{f}^{\left(n+\frac{1}{2}\right)}-\tilde{\theta}\mathbf{f}^{(n+\theta)}-\theta\mathbf{f}^{(n+1)}\right\|^2+\frac{1}{4}\left\|\mathbf{f}^{\left(n+\frac{1}{2}-\theta\right)}-\tilde{\theta}\mathbf{f}^{(n+\theta)}-\theta\mathbf{f}^{(n)}\right\|^2 \\
& +\frac{1}{4}\left\|\mathbf{f}^{\left(n+\theta-\frac{1}{2}\right)}-\theta\mathbf{f}^{(n+\theta)}-(1-2\theta)\,\mathbf{f}^{(n+\theta-1)}-\theta\mathbf{f}^{(n)}\right\|^2, \qquad (6.73)
\end{aligned}
$$

and

$$
H_p := \frac{\acute{d}^{\frac{1}{2}}}{4}\left(\left\|p^{\left(n+\frac{1}{2}\right)}-\mathcal{P}^{\left(n+\frac{1}{2}\right)}\right\|^2+\left\|p^{\left(n+\frac{1}{2}-\theta\right)}-\mathcal{P}^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2+\left\|p^{\left(n+\theta-\frac{1}{2}\right)}-\mathcal{P}^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2\right). \qquad (6.74)
$$

Multiplying expression (6.70) by $\frac{2\Delta t}{Re}$ and summing from time $n=0$ to l gives

$$
\begin{aligned}
& \left\|\boldsymbol{E}^{(l+1)}\right\|^2-\left\|\boldsymbol{E}^{(0)}\right\|^2+\left\|\boldsymbol{E}^{(l+\tilde{\theta})}\right\|^2-\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2+\left\|\boldsymbol{E}^{(l+\theta)}\right\|^2-\left\|\boldsymbol{E}^{(\theta)}\right\|^2 \\
& +\left(4\tilde{\alpha}\theta-\epsilon_2\left(2\tilde{\alpha}\theta+4\tilde{\alpha}+8\tilde{\alpha}\tilde{\theta}+2\right)\right)\frac{\Delta t}{Re}\sum_{n=0}^{l}\left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2 \\
& +\left(4\tilde{\alpha}\theta-\epsilon_3\left(4\tilde{\alpha}(1-\theta)+12\tilde{\alpha}\theta+8\tilde{\alpha}\,(1-2\theta)+4\tilde{\alpha}+2\right)\right)\frac{\Delta t}{Re}\sum_{n=0}^{l}\left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2 \\
\leq\; & \Delta t\sum_{n=0}^{l}\left\|\boldsymbol{E}^{(n+1)}\right\|^2\frac{2}{Re}\left((2Re+1)+Ch^{-2}\right)+\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{E}^{(n)}\right\|^2\frac{2Ch^{-2}}{Re}\left(\theta\tilde{\alpha}+\frac{\tilde{\alpha}\theta}{2\epsilon_3}\right) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{E}^{(n+\tilde{\theta})}\right\|^2\frac{2}{Re}\left((2Re+1)+Ch^{-2}\left(2+\frac{6\tilde{\alpha}}{2}+\frac{(\theta)2\tilde{\alpha}}{2}+\frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3}\right)\right) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{E}^{(n+\theta)}\right\|^2\frac{2}{Re}\left((2Re+1)+Ch^{-2}\left(1+\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\right)\right) \\
& +\Delta t\sum_{n=0}^{l}\left\|\boldsymbol{E}^{(n+\theta-1)}\right\|^2\frac{2Ch^{-2}\tilde{\alpha}\,(1-2\theta)}{2Re\epsilon_3} \\
& +\Delta t\sum_{n=0}^{l}\frac{2}{Re}H_{\boldsymbol{\Lambda}}+\Delta t\sum_{n=0}^{l}\frac{2}{Re}H_{\mathcal{I}^n}+\Delta t\sum_{n=0}^{l}\frac{2}{Re}H_{\mathbf{f}}+\Delta t\sum_{n=0}^{l}\frac{2}{Re}H_p. \qquad (6.75)
\end{aligned}
$$

In order to apply the discrete Gronwall's lemma we need the inequalities stated in (6.76), (6.77), and (6.78) to hold.

$$
1 \;\geq\; \frac{2\Delta t}{Re}\left(\left((2Re+1)+Ch^{-2}\right)+Ch^{-2}\left(\theta\tilde{\alpha}+\frac{\tilde{\alpha}\theta}{2\epsilon_3}\right)\right), \qquad (6.76)
$$

$$
1 \;\geq\; \frac{2\Delta t}{Re}\left((2Re+1)+Ch^{-2}\left(2+\frac{6\tilde{\alpha}}{2}+\frac{(\theta)2\tilde{\alpha}}{2}+\frac{(1-\theta)2\tilde{\alpha}}{4\epsilon_3}\right)\right), \qquad (6.77)
$$

$$
1 \;\geq\; \frac{2\Delta t}{Re}\left((2Re+1)+Ch^{-2}\left(1+\frac{2\tilde{\alpha}\tilde{\theta}}{4\epsilon_2}\right)+\frac{Ch^{-2}\tilde{\alpha}\,(1-2\theta)}{2\epsilon_3}\right). \qquad (6.78)
$$

Summarizing (6.76), (6.77), and (6.78) yields that

$$
\Delta t\left(C+Ch^{-2}\right)\leq 1.
$$

This gives the stability condition

$$
\Delta t \leq Ch^2. \qquad (6.79)
$$

6.7 The H terms in (6.75)

Using interpolation properties, and elements of order k, and q for velocity and pressure respectively, bounds are found for the $H_{\mathbf{\Lambda}}, H_{\mathcal{I}^n}, H_{\mathbf{f}}$, and H_p expressions in (6.75). Here

$$\sum_{n=0}^{l} \frac{2\Delta t}{Re} H_{\mathbf{\Lambda}} \quad \leq \quad Ch^{2k+2} \|\boldsymbol{u}_t\|_{0,k+1}^2 + Ch^{2k} \,|||\boldsymbol{u}|||_{0,k+1} \,. \tag{6.80}$$

Bounding,

$$\begin{aligned} \sum_{n=0}^{l} \frac{2\Delta t}{Re} H_{\mathcal{I}^n} \quad &\leq \quad C(\Delta t)^4 \|\boldsymbol{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\boldsymbol{\sigma}_{tt}\|_{0,0}^2 + C(\Delta t)^4 \|(\nabla \boldsymbol{u})_{tt}\|_{0,0}^2 \\ &\leq \quad C(\Delta t)^4 \|\boldsymbol{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|(\nabla \boldsymbol{u})_{tt}\|_{0,0}^2 + C(\Delta t)^4 C_T \\ &\leq \quad C(\Delta t)^4 \|\boldsymbol{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\boldsymbol{u}_{tt}\|_{0,1}^2 + C(\Delta t)^4 C_T. \end{aligned} \tag{6.81}$$

For the forcing function and the pressure we have:

$$\sum_{n=0}^{l} \frac{2\Delta t}{Re} H_{\mathbf{f}} \quad \leq \quad C(\Delta t)^4 \|\mathbf{f}_{tt}\|_{0,0}^2 \,, \tag{6.82}$$

and

$$\sum_{n=0}^{l} \frac{2\Delta t}{Re} H_p \leq Ch^{2q+2} \,|||p|||_{0,q+1}^2 \,. \tag{6.83}$$

Applying the discrete Gronwall's lemma to (6.75) we obtain

$$\begin{aligned} & \left\|\boldsymbol{E}^{l+1}\right\|^2 - \left\|\boldsymbol{E}^{(0)}\right\|^2 + \left\|\boldsymbol{E}^{(l+\tilde{\theta})}\right\|^2 - \left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2 + \left\|\boldsymbol{E}^{(l+\theta)}\right\|^2 - \left\|\boldsymbol{E}^{(\theta)}\right\|^2 \\ & + \left(4\tilde{\alpha}\theta - 2\epsilon_2 \left(6\tilde{\alpha} - 3\tilde{\alpha}\theta - 1\right)\right) \frac{\Delta t}{Re} \sum_{n=0}^{l} \left\|\mathbf{d}(\boldsymbol{E}^{(n+1)})\right\|^2 \\ & + \left(4\tilde{\alpha}\theta - 2\epsilon_3 \left(8\tilde{\alpha} - 4\tilde{\alpha}\theta + 1\right)\right) \frac{\Delta t}{Re} \sum_{n=0}^{l} \left\|\mathbf{d}(\boldsymbol{E}^{(n+\theta)})\right\|^2 \\ \leq \quad & G_{\boldsymbol{u}}(\Delta t, h, \delta), \end{aligned} \tag{6.84}$$

where

$$\begin{aligned} G_{\boldsymbol{u}}(\Delta t, h, \delta) \quad := \quad & Ch^{2k+2} \|\boldsymbol{u}_t\|_{0,k+1}^2 + Ch^{2k} \,|||\boldsymbol{u}|||_{0,k+1} + Ch^{2q+2} \,|||p|||_{0,q+1}^2 \\ & + C(\Delta t)^4 \|\boldsymbol{u}_{ttt}\|_{0,0}^2 + C(\Delta t)^4 \|\boldsymbol{u}_{tt}\|_{0,1}^2 + C(\Delta t)^4 \|\mathbf{f}_{tt}\|_{0,0}^2 \\ & + C(\Delta t)^4 C_T. \end{aligned} \tag{6.85}$$

Applying lemmas 5 and A.3 for bounds on $\left\|\boldsymbol{E}^{(\theta)}\right\|^2$ and $\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2$ respectivly, and assuming $\left\|\boldsymbol{E}^{(}0)\right\|^2 = 0$, then from (6.84) we have (6.86). Lemma 5 and lemma A.3 are stated and proved in Appendix A.

$$\|\boldsymbol{E}\|^2 + C \,|||\mathbf{d}(\boldsymbol{E})|||_{0,0}^2 \leq G_{\boldsymbol{u}}(\Delta t, h, \delta), \tag{6.86}$$

which gives

$$|||\boldsymbol{E}|||_{\infty,0}^2 \leq G_{\boldsymbol{u}}(\Delta t, h, \delta), \tag{6.87}$$

and

$$|||\boldsymbol{E}|||_{0,0}^2 \leq TG_{\boldsymbol{u}}(\Delta t, h, \delta). \tag{6.88}$$

Using Korn's lemma with (6.86) gives

$$|||\nabla \boldsymbol{E}|||_{0,0}^2 \leq CG_{\boldsymbol{u}}(\Delta t, h, \delta). \tag{6.89}$$

Hence

$$|||\boldsymbol{E}|||_{0,1}^2 \leq CG_{\boldsymbol{u}}(\Delta t, h, \delta). \tag{6.90}$$

To establish the error estimate in Theorem 4.2 note that

$$\begin{aligned} |||\boldsymbol{u} - \boldsymbol{u}_h|||_{\infty,0}^2 &\leq |||\boldsymbol{E}|||_{\infty,0}^2 + |||\Lambda|||_{\infty,0}^2 \\ &\leq G_{\boldsymbol{u}}(\Delta t, h, \delta) + Ch^{2k+2} \, |||\boldsymbol{u}|||_{\infty,k+1}^2 \, , \end{aligned} \tag{6.91}$$

and

$$\begin{aligned} |||\boldsymbol{u} - \boldsymbol{u}_h|||_{0,1}^2 &\leq |||\boldsymbol{E}|||_{0,1}^2 + |||\Lambda|||_{0,1}^2 \\ &\leq CG_{\boldsymbol{u}}(\Delta t, h, \delta) + Ch^{2k} \, |||\boldsymbol{u}|||_{0,k+1}^2 \, . \end{aligned} \tag{6.92}$$

Justification of (IH1) for $\tilde{\boldsymbol{u}}_h$.

Assume that (IH1) holds for $n = 1, 2, \ldots, l-1$. Then using inverse estimates, interpolation properties, and (4.15)

$$\begin{aligned} \left\|\boldsymbol{u}_h^l\right\|_\infty &\leq \left\|\boldsymbol{u}_h^l - \boldsymbol{u}^l\right\|_\infty + \left\|\boldsymbol{u}^l\right\|_\infty \\ &\leq \left\|\boldsymbol{E}^l\right\|_\infty + \left\|\Lambda^l\right\|_\infty + M \\ &\leq Ch^{\frac{-\acute{d}}{2}} \left\|\boldsymbol{E}^l\right\| + Ch^{\frac{-\acute{d}}{2}} \left\|\Lambda^l\right\| + M \\ &\leq C\left(h^{k-\frac{\acute{d}}{2}} + h^{q-\frac{\acute{d}}{2}+1} + (\Delta t)^2 h^{\frac{-\acute{d}}{2}} + h^{k-\frac{\acute{d}}{2}+1}\right) + M. \end{aligned} \tag{6.93}$$

Setting $k \geq \frac{\acute{d}}{2}$, and $q \geq \frac{\acute{d}}{2} - 1$ and choosing h, and Δt such that

$$h^{k-\frac{\acute{d}}{2}}, h^{q-\frac{\acute{d}}{2}+1} \leq \frac{1}{C}, \quad \text{and} \quad \Delta t^2 \leq \frac{h^{\frac{\acute{d}}{2}}}{C}, \tag{6.94}$$

we have

$$\left\|\boldsymbol{u}_h^l\right\|_\infty \leq M + 4.$$

Similarly it follows that $\left\|\boldsymbol{u}_h^{(n+\tilde{\theta})}\right\|_\infty \leq M + 4.$

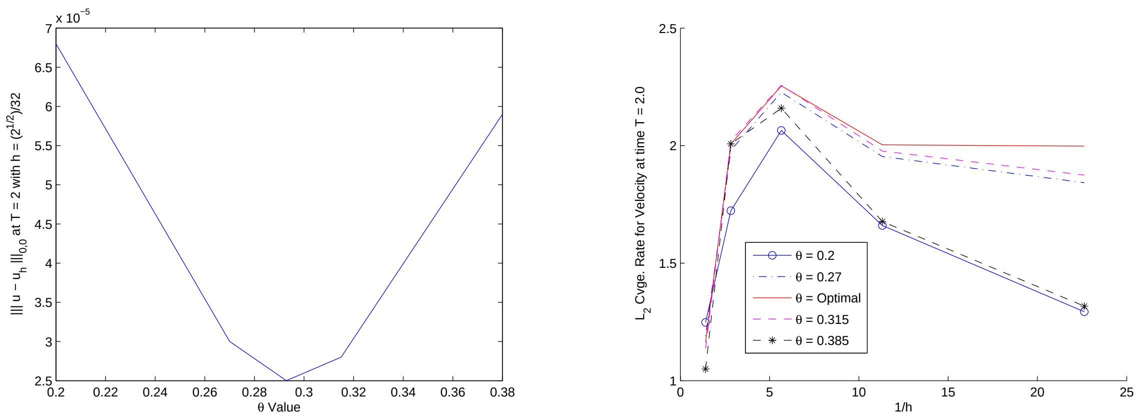
■

7 Numerical Results

In this section we present numerical results for the θ-method on two test problems. The first example uses a known analytical solution to verify numerical convergence rates (Cvge. Rate) for

the θ-method. In the second example, we considered a prototypal problem of viscoelastic flow, flow through a 4:1 planar contraction. In both examples finite element computations were carried out using the FreeFem++ integrated development environment [13]. Continuous piecewise quadratic elements were used for modeling the velocity, and continuous piecewise linear elements were used for the pressure and stress. The constitutive equation was stabilized using a SUPG discretization with a parameter δ.

For the (optimal) value of $\theta = 1 - \sqrt{2}/2 \approx 0.29289$ the local temporal discretization errors are $O((\Delta t)^2)$. The influence of the value for θ on the numerical approximations is illustrated in Figure 7.1. for computations performed on Example 1 (described below). All the other computations reported in this paper were obtained using $\theta = 1 - \sqrt{2}/2$. The constitutive equation splitting parameter ω was set to 1/2 in all computations.



(a) Error $|||\boldsymbol{u} - \boldsymbol{u}_h|||_{0,0}$ as a function of θ.

(b) Experimental convergence rates.

Figure 7.1: Optimal θ value

7.1 Example 1

In order to investigate the predicted convergence rates we consider fluid flow across a unit square with a known solution.

Let $\Omega = (0,1) \times (0,1)$, $Re = 1, \alpha = 1/2, \lambda = 2,$ and $a = 1$. For the true solution we use

$$\boldsymbol{u} = \begin{pmatrix} e^{\left(x+y-\frac{1}{2}t\right)}(x^2-x)(y^2-y) \\ -e^{(x+y-t)}(x^2-x)(y^2-y) \end{pmatrix}, \tag{7.1}$$

$$p = \cos(2\pi x)(y^2 - y), \tag{7.2}$$

$$\boldsymbol{\sigma} = 2\alpha \mathbf{d}(\boldsymbol{u}). \tag{7.3}$$

Remark: A right-hand-side function is added to (2.1) and $\mathbf{f}$ in (2.2) is calculated using (7.1)-(7.3).

For this example three sequences of computations were performed:

(i) approximation of $\tilde{\boldsymbol{\sigma}}_h$, assuming $\boldsymbol{u}$ and p (Theorem 4.1),

(ii) approximation of $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}_h$, assuming $\boldsymbol{\sigma}$ (Theorem 4.2),

(iii) approximation of $\boldsymbol{u}_h, p_h$ and $\boldsymbol{\sigma}_h$.

7.1.1 Approximating $\tilde{\boldsymbol{\sigma}}_h$ with $\boldsymbol{u}$ and p known

We first consider the approximation of the stress $\tilde{\boldsymbol{\sigma}}_h$ assuming the velocity and pressure functions are known. This is analogous to implementing Step 1a, Step 2b, and Step 3a of the θ-method. From Corollary 1 we have predicted asymptotic convergence rate

$$|||\boldsymbol{u} \cdot \nabla (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)|||_{\tilde{\theta}} \leq C\left((\Delta t)^2 + \Delta t\delta + h\delta + h + \delta\right),$$

which is consistent with the numerical convergence rates presented in Table 7.1. Note Table 7.1 shows the effect of the upwinding parameter δ on $|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0}$.

For the proof of Theorem 4.1 we required the restriction $\Delta t \leq Ch^2$. The numerical computations were performed with $\Delta t \sim h$. It is an open question if the restriction $\Delta t \leq Ch^2$ is a necessary condition for (4.12).

Table 7.1: Approximation errors and experimental convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(1, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$
0	$\lvert\lvert\lvert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\rvert\rvert\rvert_{0,0}$	2.1235e-1	6.6773e-2	1.9191e-2	5.0437e-3	1.2830e-3
	Cvge. Rate	-	1.7	1.8	1.9	2.0
	$\lvert\lvert\lvert\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\rvert\rvert\rvert_{\tilde{\theta}}$	1.1544e-1	6.1759e-2	2.9979e-2	1.5185e-2	7.6944e-3
	Cvge. Rate	-	0.9	1.0	1.0	1.0
$\frac{h}{\sqrt{2}}$	$\lvert\lvert\lvert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\rvert\rvert\rvert_{0,0}$	2.0070e-1	8.4563e-2	3.7449e-2	1.6980e-2	8.0428e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.1
	$\lvert\lvert\lvert\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\rvert\rvert\rvert_{\tilde{\theta}}$	1.0263e-1	6.4336e-2	3.2570e-2	1.6325e-2	8.2071e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$\lvert\lvert\lvert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\rvert\rvert\rvert_{0,0}$	2.0174e-1	7.3678e-2	2.3575e-2	6.9629e-3	2.0645e-3
	Cvge. Rate	-	1.5	1.6	1.8	1.8
	$\lvert\lvert\lvert\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\rvert\rvert\rvert_{\tilde{\theta}}$	1.0447e-1	6.2239e-2	3.0497e-2	1.5304e-2	7.7199e-3
	Cvge. Rate	-	0.7	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{2}$	$\lvert\lvert\lvert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\rvert\rvert\rvert^2_{0,0}$	2.0346e-1	6.9501e-2	2.0245e-2	5.3281e-3	1.3546e-3
	Cvge. Rate	-	1.5	1.8	1.9	2.0
	$\lvert\lvert\lvert\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h)\rvert\rvert\rvert_{\tilde{\theta}}$	1.0664e-1	6.1737e-2	3.0104e-2	1.5203e-2	7.6968e-3
	Cvge. Rate	-	0.8	1.0	1.0	1.0

7.1.2 Approximating $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}_h$ with known $\boldsymbol{\sigma}$

Numerical results for the approximation of velocity, $\tilde{\boldsymbol{u}}_h$, and pressure, $\tilde{p}_h$ for a known stress, $\boldsymbol{\sigma}$, are presented in Table 7.2. These results correspond to the analysis of step 1b, step 2a, and step 3b as stated in Theorem 4.2.

The numerical convergence rates observed are consistent with those predicted in Corollary 2 where

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} \leq O\left((\Delta t)^2 + h^2\right).$$

Similar to the case in the previous subsection we make note that the proof of Theorem 4.2 required the restriction $\Delta t \leq Ch^2$. Here the numerical computations were performed with $\Delta t \sim h$.

Table 7.2: Approximation errors and experimental convergence rates at $T = 2$

$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$	$\left(\frac{1}{64}, \frac{\sqrt{2}}{128}\right)$
$\vert\vert\vert \boldsymbol{u} - \tilde{\boldsymbol{u}}_h \vert\vert\vert_{0,1}$	4.4196e-2	1.1485e-2	2.9707e-3	7.5759e-4	1.9142e-4	4.8129e-5
Cvge. Rate	-	1.9	2.0	2.0	2.0	2.0
$\vert\vert\vert \boldsymbol{u} - \tilde{\boldsymbol{u}}_h \vert\vert\vert_{\infty,0}$	1.4734e-3	1.7996e-4	2.5014e-5	4.1759e-6	8.5692e-7	2.1636e-7
Cvge. Rate	-	3.0	2.8	2.6	2.3	2.0
$\vert\vert\vert p - \tilde{p}_h \vert\vert\vert_{0,0}$	1.0859e-1	6.6842e-3	1.5033e-3	3.9097e-4	1.2086e-4	4.7884e-5
Cvge. Rate	-	4.0	2.2	1.9	1.7	1.3
$\vert\vert\vert p - \tilde{p}_h \vert\vert\vert_{\infty,0}$	8.4003e-2	4.9703e-3	1.1343e-3	3.2878e-4	1.2797e-4	6.0659e-5
Cvge. Rate	-	4.1	2.1	1.8	1.4	1.1

7.1.3 Full θ-method approximation for viscoelasticity

Table 7.3 contains the results for the approximation of $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ using the θ-method described in Step 1a - Step 3b. The numerical convergence rates are consistent with our expectations based on Theorems 4.1 and 4.2, i.e.

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{0,1} + |||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h|||_{0,0} + |||\boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||_{\tilde{\theta}} \sim O\left((\Delta t)^2 + (\Delta t)\,\delta + \delta + h\right).$$

7.2 Example 2

For a second example the numerical approximation of viscoelastic flow through a planar 4:1 contraction channel is presented. This has been a long standing benchmark problem for viscoelastic flow [16, 19, 20]. A diagram of the flow geometry is given in Figure 7.2. It is assumed that the channel lengths are sufficiently long for fully developed Poiseuille flow at both the inflow and outflow boundaries. In the computations the value of L in Figure 7.2 is set at 1/4.

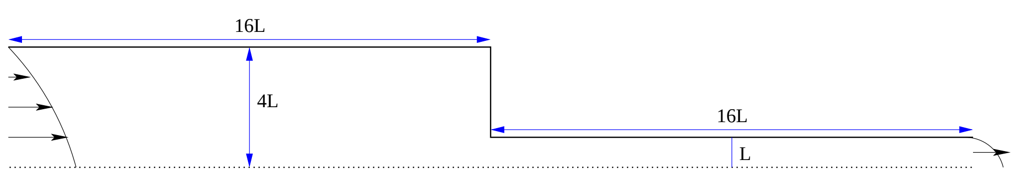


Figure 7.2: Plot of 4:1 contraction domain geometry.

The flow at $t = 0$ is assumed to be stationary and then slowly increased for $t > 0$ using $A(t) = 1 - e^{-t}$. The boundary conditions at the inflow of the channel are defined by

$$\boldsymbol{u} = A(t) \begin{pmatrix} \frac{1}{32}\left(1 - y^2\right) \\ 0 \end{pmatrix}, \tag{7.4}$$

Table 7.3: Approximation errors and experimental convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$
0	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.7608e-2	1.2323e-2	3.2034e-3	8.3187e-4	2.1793e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	6.5569e-2	1.9248e-2	5.0845e-3	1.3000e-3	3.2981e-4
	Cvge. Rate	-	1.8	1.9	2.0	2.0
	$\lvert\lvert\lvert \boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \rvert\rvert\rvert_{\tilde{\theta}}$	6.1217e-2	2.9982e-2	1.5191e-2	7.6984e-3	3.8800e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\frac{h}{\sqrt{2}}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	5.0150e-2	1.6193e-2	6.4636e-3	2.9147e-3	1.3989e-3
	Cvge. Rate	-	1.6	1.3	1.1	1.1
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	8.1922e-2	3.5905e-2	1.5791e-2	7.3743e-3	3.5789e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.0
	$\lvert\lvert\lvert \boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \rvert\rvert\rvert_{\tilde{\theta}}$	6.4087e-2	3.2589e-2	1.6309e-2	8.2007e-3	4.1180e-4
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.8620e-2	1.3135e-2	3.6334e-3	1.0192e-3	2.9513e-4
	Cvge. Rate	-	1.9	1.9	1.8	1.8
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	7.1941e-2	2.3392e-2	6.8309e-3	1.9976e-3	6.0598e-4
	Cvge. Rate	-	1.6	1.8	1.8	1.7
	$\lvert\lvert\lvert \boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \rvert\rvert\rvert_{\tilde{\theta}}$	6.1860e-2	3.0510e-2	1.5305e-2	7.7221e-3	3.8851e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{2}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.8022e-2	1.2507e-2	3.2644e-3	8.4818e-4	2.2193e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	6.8104e-2	2.0308e-2	5.3614e-3	1.3680e-3	3.4645e-4
	Cvge. Rate	-	1.7	1.9	2.0	2.0
	$\lvert\lvert\lvert \boldsymbol{u} \cdot \nabla(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \rvert\rvert\rvert_{\tilde{\theta}}$	6.1279e-2	3.0111e-2	1.5208e-2	7.7004e-3	3.8803e-3
	Cvge. Rate	-	1.0	1.0	1.0	1.0

and

$$\sigma_{11} = \frac{\lambda A(t)^2 y^2 \alpha(1+a)}{D(t)}, \quad \sigma_{12} = \frac{-16\alpha A(t) y}{D(t)}, \quad \text{and} \quad \sigma_{22} = \frac{\lambda A(t)^2 y^2 \alpha(a-1)}{D(t)}, \tag{7.5}$$

where

$$D(t) = 256 + A(t)^2 y^2 \lambda^2 (1+a)(1-a).$$

The outflow boundary condition is

$$\boldsymbol{u} = A(t) \begin{pmatrix} 2\left(\frac{1}{16} - y^2\right) \\ 0 \end{pmatrix}. \tag{7.6}$$

No slip boundary conditions were imposed for the velocity on the solid walls of the contraction, and a symmetry condition was imposed along the bottom of the computational domain. The computations were performed on a uniformly refined version of the mesh shown in Figure 7.3 with $\Delta x_{\min} = 0.0625$ and $\Delta y_{\min} = 0.015625$.

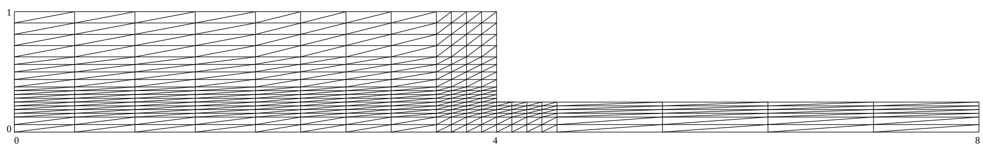
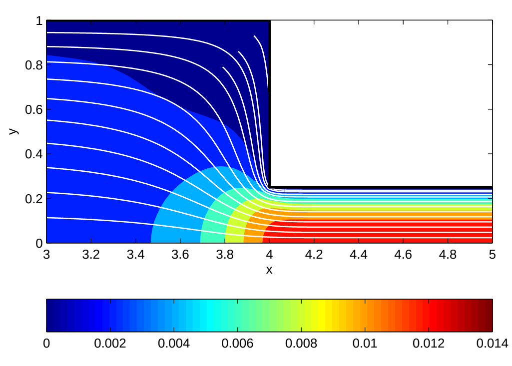
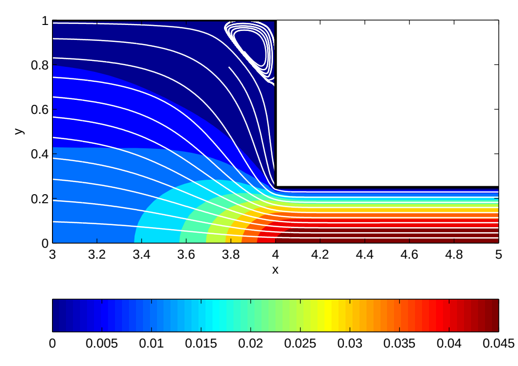


Figure 7.3: Sample Contraction Mesh

The computations were done using the full θ-method approximation given by (4.2) - (4.7) for an Oldroyd B fluid ($a = 1$), with $\lambda = 2$ and $Re = 1$. The value of α was set to 8/9, which is commonly used in the literature [19]. The discrete time step and upwinding parameter were set to $\Delta t = 1/32$ and $\delta = (2/\Delta y_{\min})^2$. Figures 7.4 and 7.5 show streamlines for the fluid at times $t = 1/8, 1/2, 1,$ and 4 superimposed on a contour plot showing the magnitude of velocity. Note that, consistent with expectations, as the velocity is increased, a vortex appears in the upper corner of the domain and grows with the magnitude of the velocity.

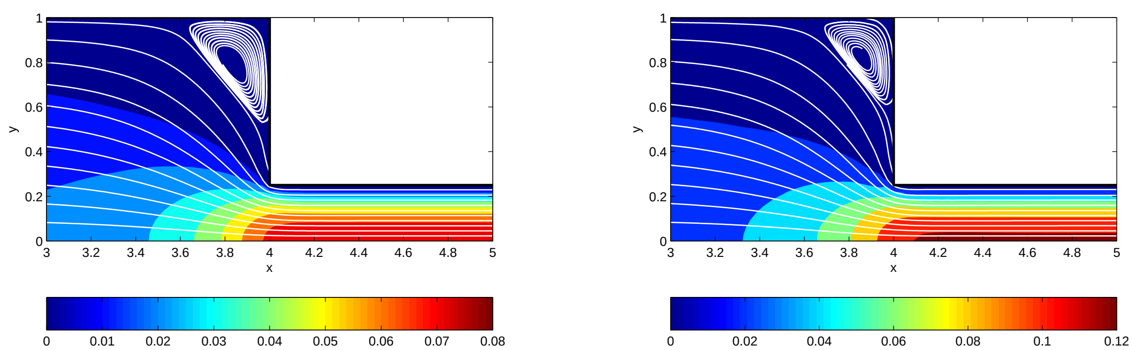


(a) $t = 0.125$



(b) $t = 0.5$

Figure 7.4: Streamlines and magnitude contours for $\boldsymbol{u}$



(a) $t = 1$

(b) $t = 4$

Figure 7.5: Streamline and magnitude contours for $\boldsymbol{u}$

A Initial Bounds

In this section we establish error estimates for $\left\|\boldsymbol{F}^{(\theta)}\right\|^2$, $\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2$, $\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2$, $\left\|\boldsymbol{E}^{(\theta)}\right\|^2$, and $\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2$ used in the analysis of the θ-method.

A.1 $\left\|\boldsymbol{F}^{(\theta)}\right\|^2$, $\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2$, and $\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2$

Lemma A.1

$$\begin{aligned}\left\|\boldsymbol{F}^{(\theta)}\right\|^2 \;\le\;& Ch^{2m+2}+C\Delta t^2h^{2m+2}+C\delta^2\Delta t^2h^{2m}\\ &+C\Delta t^2h^{2m}+C\delta^2\Delta t^2h^{2m-2}+C\Delta t^4+C\delta^2\Delta t^2h^{-2}+C\Delta t^4\delta^2h^{-2}. \qquad (A.1)\end{aligned}$$

Proof: Subtracting ((4.2) with known $\boldsymbol{u}$ and $n=0$) from ((3.2) at $t=0$):

$$\begin{aligned}\frac{\lambda}{\theta\Delta t}\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)},\boldsymbol{\tau}\right)+\omega\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)},\boldsymbol{\tau}_{\delta^{(0)}}\right)=&\frac{\lambda}{\theta\Delta t}\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(0)},\boldsymbol{\tau}\right)-(1-\omega)\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(0)},\boldsymbol{\tau}_{\delta^{(0)}}\right)-\lambda\left(\boldsymbol{u}^{(n)}\cdot\nabla\mathbf{e}_{\boldsymbol{\sigma}}^{(0)},\boldsymbol{\tau}_{\delta^{(0)}}\right)\\&-\lambda\left(g_a(\mathbf{e}_{\boldsymbol{\sigma}}^{(0)},\nabla\boldsymbol{u}^{(0)}),\boldsymbol{\tau}_{\delta^{(0)}}\right)+\lambda\left(\frac{\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)}}{\theta\Delta t}-\boldsymbol{\sigma}_t^{(0)},\boldsymbol{\tau}\right)\\&+\omega\left(\boldsymbol{\sigma}^{(0)}-\boldsymbol{\sigma}^{(\theta)},\boldsymbol{\tau}_{\delta^{(0)}}\right)+\lambda\left(-\boldsymbol{\sigma}_t^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{\tau}\right). \qquad (A.2)\end{aligned}$$

Recall that $\mathbf{e}_{\boldsymbol{\sigma}}^{(\mu)}=\boldsymbol{\Gamma}^{(\mu)}+\boldsymbol{F}^{(\mu)}$. Let $\boldsymbol{\tau}=\boldsymbol{F}^{(\theta)}$ and (A.2) yields

$$\begin{aligned}&\left\|\boldsymbol{F}^{(\theta)}\right\|^2+\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\theta)}\right)+\frac{\omega\theta\Delta t}{\lambda}\left\|\boldsymbol{F}^{(\theta)}\right\|^2+\frac{\omega\theta\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\theta)}\right)\\&\quad+\frac{\omega\theta\Delta t}{\lambda}\left(\boldsymbol{F}^{(\theta)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)+\frac{\omega\theta\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\ =&\;\left(\boldsymbol{F}^{(0)},\boldsymbol{F}^{(\theta)}\right)+\left(\boldsymbol{\Gamma}^{(0)},\boldsymbol{F}^{(\theta)}\right)\\&\quad-\frac{\tilde{\omega}\theta\Delta t}{\lambda}\left(\boldsymbol{F}^{(0)},\boldsymbol{F}^{(\theta)}\right)-\frac{\tilde{\omega}\theta\Delta t}{\lambda}\left(\boldsymbol{F}^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\frac{\tilde{\omega}\theta\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(0)},\boldsymbol{F}^{(\theta)}\right)-\frac{\tilde{\omega}\theta\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\theta\Delta t\left(\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(0)},\boldsymbol{F}^{(\theta)}\right)-\theta\Delta t\left(\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\theta\Delta t\left(\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{\Gamma}^{(0)},\boldsymbol{F}^{(\theta)}\right)-\theta\Delta t\left(\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{\Gamma}^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\theta\Delta t\left(g_a(\boldsymbol{F}^{(0)},\nabla\boldsymbol{u}^{(0)}),\boldsymbol{F}^{(\theta)}\right)-\theta\Delta t\left(g_a(\boldsymbol{F}^{(0)},\nabla\boldsymbol{u}^{(0)}),\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\theta\Delta t\left(g_a(\boldsymbol{\Gamma}^{(0)},\nabla\boldsymbol{u}^{(0)}),\boldsymbol{F}^{(\theta)}\right)-\theta\Delta t\left(g_a(\boldsymbol{\Gamma}^{(0)},\nabla\boldsymbol{u}^{(0)}),\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad-\theta\Delta t\omega\left(\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)},\boldsymbol{F}^{(\theta)}\right)-\theta\Delta t\omega\left(\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)},\delta\boldsymbol{u}^{(0)}\nabla\boldsymbol{F}^{(\theta)}\right)\\&\quad+\theta\Delta t\left(\frac{\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)}}{\theta\Delta t}-\boldsymbol{\sigma}_t^{(0)},\boldsymbol{F}^{(\theta)}\right)+\theta\Delta t\left(-\boldsymbol{\sigma}_t^{(0)},\delta\boldsymbol{u}^{(0)}\cdot\nabla\boldsymbol{F}^{(\theta)}\right). \qquad (A.3)\end{aligned}$$

Appropriate bounds can now be placed on each of the terms in (A.3). Thus,

$$\left(\boldsymbol{\Gamma}^{(\theta)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{1}{4\epsilon_1} \left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^2 + \epsilon_1 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.4}$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\theta)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{\omega^2\theta^2(\Delta t)^2}{4\epsilon_2\lambda^2} \left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^2 + \epsilon_2 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.5}$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\boldsymbol{F}^{(\theta)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.6}$$

$$\frac{\omega\theta\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(\theta)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \acute{d} C h^{-2}\omega^2\theta^2(\Delta t)^2}{4\epsilon_3\lambda^2} \left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^2 + \epsilon_3 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.7}$$

$$\left(\boldsymbol{F}^{(0)}, \boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.8}$$

$$\left(\boldsymbol{\Gamma}^{(0)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{1}{4\epsilon_4} \left\|\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_4 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.9}$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\boldsymbol{F}^{(0)}, \boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.10}$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\boldsymbol{F}^{(0)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.11}$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(0)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{\tilde{\omega}^2\theta^2(\Delta t)^2}{4\lambda^2\epsilon_5} \left\|\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_5 \left\|\boldsymbol{F}^{(\theta)}\right\|^2 \tag{A.12}$$

$$\frac{\tilde{\omega}\theta\Delta t}{\lambda} \left(\boldsymbol{\Gamma}^{(0)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \acute{d}\tilde{\omega}^2\theta^2(\Delta t)^2 C h^{-2}}{4\lambda^2\epsilon_6} \left\|\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_6 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.13}$$

$$\theta\Delta t \left(\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(0)}, \boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.14}$$

$$\theta\Delta t \left(\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(0)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.15}$$

$$\theta\Delta t \left(\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{\Gamma}^{(0)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{M^2 \acute{d}\theta^2(\Delta t)^2}{4\epsilon_7} \left\|\nabla\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_7 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.16}$$

$$\theta\Delta t \left(\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{\Gamma}^{(0)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) \leq \frac{\delta^2 \acute{d}^2 M^4\theta^2(\Delta t)^2 C h^{-2}}{4\epsilon_8} \left\|\nabla\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_8 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.17}$$

$$\theta\Delta t \left(g_a(\boldsymbol{F}^{(0)}, \nabla\boldsymbol{u}^{(0)}), \boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.18}$$

$$\theta\Delta t \left(g_a(\boldsymbol{F}^{(0)}, \nabla\boldsymbol{u}^{(0)}), \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) = 0, \tag{A.19}$$

$$\theta\Delta t \left(g_a(\boldsymbol{\Gamma}^{(0)}, \nabla\boldsymbol{u}^{(0)}), \boldsymbol{F}^{(\theta)}\right) \leq \frac{4M^2 \acute{d}^2\theta^2(\Delta t)^2}{\epsilon_9} \left\|\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_9 \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.20}$$

$$\theta\Delta t \left(g_a(\boldsymbol{\Gamma}^{(0)}, \nabla\boldsymbol{u}^{(0)}), \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) \leq \frac{\delta^2 4M^2 \acute{d}^3\theta^2(\Delta t)^2 C h^{-2}}{\epsilon_{10}} \left\|\boldsymbol{\Gamma}^{(0)}\right\|^2 + \epsilon_{10} \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.21}$$

$$\theta\Delta t \left(\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta\Delta t} - \boldsymbol{\sigma}_t^{(0)}, \boldsymbol{F}^{(\theta)}\right) \leq \frac{\theta^2(\Delta t)^2}{4\epsilon_{11}} \left\|\frac{\boldsymbol{\sigma}^{(\theta)} - \boldsymbol{\sigma}^{(0)}}{\theta\Delta t} - \boldsymbol{\sigma}_t^{(0)}\right\|^2 + \epsilon_{11} \left\|\boldsymbol{F}^{(\theta)}\right\|^2, \tag{A.22}$$

$$\theta\Delta t \left(-\boldsymbol{\sigma}_t^{(0)}, \delta\boldsymbol{u}^{(0)} \cdot \nabla\boldsymbol{F}^{(\theta)}\right) \leq \frac{\delta^2 M^2 \acute{d}\theta^2(\Delta t)^2 C h^{-2}}{4\epsilon_{12}} \left\|\boldsymbol{\sigma}_t^{(0)}\right\|^2 + \epsilon_{12} \left\|\boldsymbol{F}^{(\theta)}\right\|^2. \tag{A.23}$$

$$\theta\Delta t\omega\left(\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)},\boldsymbol{F}^{(\theta)}\right)\leq\frac{\theta^2(\Delta t)^2\omega^2}{4\epsilon_{13}}\left\|\boldsymbol{u}^{(\theta)}-\boldsymbol{u}^{(0)}\right\|^2+\epsilon_{13}\left\|\boldsymbol{F}^{(\theta)}\right\|^2 \tag{A.24}$$

$$\theta\Delta t\omega\left(\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)},\delta\boldsymbol{u}^{(0)}\nabla\boldsymbol{F}^{(\theta)}\right)\leq\frac{\delta^2\theta^2(\Delta t)^2\omega^2\acute{d}M^2Ch^{-2}}{4\epsilon_{14}}\left\|\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)}\right\|^2+\epsilon_{14}\left\|\boldsymbol{F}^{(\theta)}\right\|^2 \tag{A.25}$$

Using (A.4) - (A.25) in (A.3) we obtain:

$$\begin{aligned}
&\left\|\boldsymbol{F}^{(\theta)}\right\|^2\left(1+\frac{\omega\theta\Delta t}{\lambda}-\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5-\epsilon_6-\epsilon_7-\epsilon_8-\epsilon_9-\epsilon_{10}-\epsilon_{11}-\epsilon_{12}-\epsilon_{13}-\epsilon_{14}\right)\\
\leq\ &\left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^2\left(\frac{1}{4\epsilon_1}+\frac{\omega^2\theta^2\Delta t^2}{4\lambda^2\epsilon_2}+\frac{\delta^2M^2\acute{d}Ch^{-2}\omega^2\theta^2\Delta t^2}{4\epsilon_3\lambda^2}\right)\\
&+\left\|\boldsymbol{\Gamma}^{(0)}\right\|^2\left(\frac{1}{4\epsilon_4}+\frac{\tilde{\omega}^2\theta^2\Delta t^2}{4\epsilon_5\lambda^2}+\frac{\delta^2M^2\acute{d}Ch^{-2}\tilde{\omega}^2\theta^2\Delta t^2}{4\epsilon_6\lambda^2}+\frac{4M^2\acute{d}^2\theta^2\Delta t^2}{\epsilon_9}+\frac{4M^2\acute{d}^3Ch^{-2}\delta^2\theta^2\Delta t^2}{\epsilon_{10}}\right)\\
&+\left\|\nabla\boldsymbol{\Gamma}^{(0)}\right\|^2\left(\frac{M^2\acute{d}\theta^2\Delta t^2}{4\epsilon_7}+\frac{\acute{d}^2M^4\delta^2Ch^{-2}\theta^2\Delta t^2}{4\epsilon_8}\right)\\
&+\left\|\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)}\right\|^2\left(\frac{\theta^2(\Delta t)^2\omega^2}{4\epsilon_{13}}+\frac{\delta^2\theta^2\Delta t^2\omega^2\acute{d}M^2Ch^{-2}}{4\epsilon_{14}}\right)\\
&+\frac{\theta^2\Delta t^2}{4\epsilon_{11}}\left\|\frac{\boldsymbol{\sigma}^{(\theta)}-\boldsymbol{\sigma}^{(0)}}{\theta\Delta t}-\boldsymbol{\sigma}_t^{(0)}\right\|^2+\frac{\delta^2M^2\acute{d}\theta^2\Delta t^2Ch^{-2}}{4\epsilon_{12}}\left\|\boldsymbol{\sigma}_t^{(0)}\right\|^2.
\end{aligned} \tag{A.26}$$

Choose $\epsilon_i=1/24$, for $i\in\{1,\ldots,12\}$. Then using interpolation properties and lemmas C.17 and C.20 with (A.26) gives:

$$\begin{aligned}
\left\|\boldsymbol{F}^{(\theta)}\right\|^2\ \leq\ & Ch^{2m+2}\left\|\boldsymbol{\sigma}^{(\theta)}\right\|_{m+1}^2+C\Delta t^2h^{2m+2}\left\|\boldsymbol{\sigma}^{(\theta)}\right\|_{m+1}^2+C\delta^2\Delta t^2h^{2m}\left\|\boldsymbol{\sigma}^{(\theta)}\right\|_{m+1}^2\\
&+Ch^{2m+2}\left\|\boldsymbol{\sigma}^{(0)}\right\|_{m+1}^2+C\Delta t^2h^{2m+2}\left\|\boldsymbol{\sigma}^{(0)}\right\|_{m+1}^2+C\delta^2\Delta t^2h^{2m}\left\|\boldsymbol{\sigma}^{(0)}\right\|_{m+1}^2\\
&+C\Delta t^2h^{2m}\left\|\boldsymbol{\sigma}^{(\theta)}\right\|_{m+1}^2+C\Delta t^2\delta^2h^{2m-2}\left\|\boldsymbol{\sigma}^{(\theta)}\right\|_{m+1}^2+C\Delta t^4+C\delta^2\Delta t^2h^{-2}\left\|\boldsymbol{\sigma}_t^{(0)}\right\|^2\\
&+C\Delta t^4\delta^2h^{-2},
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
\Longrightarrow\left\|\boldsymbol{F}^{(\theta)}\right\|^2\ \leq\ & Ch^{2m+2}+C\Delta t^2h^{2m+2}+C\delta^2\Delta t^2h^{2m}\\
&+C\Delta t^2h^{2m}+C\delta^2\Delta t^2h^{2m-2}+C\Delta t^4\\
&+C\delta^2\Delta t^2h^{-2}+C\Delta t^4\delta^2h^{-2}.
\end{aligned} \tag{A.28}$$

■

Lemma A.2 *For sufficiently smooth solutions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, and p,*

$$\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2\leq S(h,\Delta t,\delta),\qquad\textit{and}\qquad\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2\leq S(h,\Delta t,\delta), \tag{A.29}$$

where

$$
\begin{aligned}
S(h,\Delta t,\delta) \;=\;& C\big[h^{2m+2}+\Delta t^2 h^{2m+2}+\delta^2\Delta t^2 h^{2m}+\Delta t^2 h^{2m} \\
&\qquad +\delta^2\Delta t^2 h^{2m-2}+\Delta t^4+\delta^2\Delta t^2 h^{-2}+C\Delta t^4\delta^2 h^{-2}\big]\left(1+\Delta t^2+\Delta t\delta\right) \\
&+C\left(h^{2m+2}+\Delta t^2 h^{2m+2}+\Delta t\delta h^{2m+2}\right) \\
&+C\left(\Delta t^2 h^{2m}+\Delta t\delta h^{2m}\right)+C(\Delta t)^4+C\delta^2\Delta t^2.
\end{aligned}
$$

Proof: Subtracting ((4.5) with known $\boldsymbol{u}$ and $n=0$) from ((3.2) at $t=\tilde{\theta}$):

$$
\begin{aligned}
&\frac{\lambda}{(1-2\theta)\Delta t}\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})},\boldsymbol{\tau}\right)+\tilde{\omega}\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})},\boldsymbol{\tau}_{\delta(\tilde{\theta})}\right)+\lambda\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})},\boldsymbol{\tau}_{\delta(\tilde{\theta})}\right)+\lambda\left(g_a(\mathbf{e}_{\boldsymbol{\sigma}}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\boldsymbol{\tau}_{\delta(\tilde{\theta})}\right) \\
&\qquad =\frac{\lambda}{(1-2\theta)\Delta t}\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)},\boldsymbol{\tau}\right)-\omega\left(\mathbf{e}_{\boldsymbol{\sigma}}^{(\theta)},\boldsymbol{\tau}_{\delta(\tilde{\theta})}\right)+\lambda\left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})}-\boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\,\Delta t}-\boldsymbol{\sigma}_t^{(\tilde{\theta})},\boldsymbol{\tau}\right) \\
&\qquad\qquad +\lambda\left(-\boldsymbol{\sigma}_t^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{\tau}\right)-\omega\left(\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta},\boldsymbol{\tau}_{\delta(\tilde{\theta})}\right). \qquad \text{(A.30)}
\end{aligned}
$$

Observing that $\mathbf{e}_{\boldsymbol{\sigma}}^{(\mu)}=\boldsymbol{F}^{(\mu)}+\boldsymbol{\Gamma}^{(\mu)}$ and choose $\boldsymbol{\tau}=\boldsymbol{F}^{(\tilde{\theta})}$. Then from (A.30) we have

$$
\begin{aligned}
&\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2+\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad +\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2+\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}(\tilde{\theta})\right) \\
&\quad +\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}(\tilde{\theta})\right)+\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}(\tilde{\theta})\right) \\
&\quad +(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)+\delta\,(1-2\theta)\,\Delta t\left\|\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2 \\
&\quad +(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)+(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad +(1-2\theta)\,\Delta t\left(g_a(\boldsymbol{F}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\boldsymbol{F}^{(\tilde{\theta})}\right)+(1-2\theta)\,\Delta t\left(g_a(\boldsymbol{F}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad +(1-2\theta)\,\Delta t\left(g_a(\boldsymbol{\Gamma}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\boldsymbol{F}^{(\tilde{\theta})}\right)+(1-2\theta)\,\Delta t\left(g_a(\boldsymbol{\Gamma}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
=\;&\left(\boldsymbol{F}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)+\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad -\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)-\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\theta)},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad -\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)-\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad +(1-2\theta)\,\Delta t\left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})}-\boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\,\Delta t}-\boldsymbol{\sigma}_t^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)+(1-2\theta)\,\Delta t\left(-\boldsymbol{\sigma}_t^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \\
&\quad -\omega\left(\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta},\boldsymbol{F}^{(\tilde{\theta})}+\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right). \qquad \text{(A.31)}
\end{aligned}
$$

Bounding all the terms in (A.31),

$$\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)\le\frac{1}{4\epsilon_1}\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^2+\epsilon_1\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2, \qquad \text{(A.32)}$$

$$\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}(\tilde{\theta})\right)=0, \qquad \text{(A.33)}$$

$$\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}(\tilde{\theta})\right)\leq\frac{\tilde{\omega}^{2}\,(1-2\theta)^{2}\,\Delta t^{2}}{\lambda^{2}4\epsilon_{2}}\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}+\epsilon_{2}\left\|\boldsymbol{F}(\tilde{\theta})\right\|^{2},\tag{A.34}$$

$$\begin{aligned}\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}(\tilde{\theta})\right)\quad&\leq\quad\frac{\delta\tilde{\omega}^{2}\,(1-2\theta)\,\Delta t}{\lambda^{2}4\epsilon_{3}}\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}\\&+\epsilon_{3}\delta\,(1-2\theta)\,\Delta t\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}(\tilde{\theta})\right\|^{2},\end{aligned}\tag{A.35}$$

$$+\,(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)=0,\tag{A.36}$$

$$(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{(1-2\theta)^{2}\,\Delta t^{2}M^{2}\acute{d}}{4\epsilon_{4}}\left\|\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}+\epsilon_{4}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.37}$$

$$\begin{aligned}(1-2\theta)\,\Delta t\left(\boldsymbol{u}^{\tilde{\theta}}\cdot\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right)\quad&\leq\quad\frac{\delta\,(1-2\theta)\,\Delta tM^{2}\acute{d}}{4\epsilon_{5}}\left\|\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}\\&+\epsilon_{5}\delta\,(1-2\theta)\,\Delta t\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\end{aligned}\tag{A.38}$$

$$(1-2\theta)\,\Delta t\left(g_{a}(\boldsymbol{F}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\boldsymbol{F}^{(\tilde{\theta})}\right)\leq4\,(1-2\theta)\,\Delta tM\acute{d}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.39}$$

$$\begin{aligned}&(1-2\theta)\,\Delta t\left(g_{a}(\boldsymbol{F}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right)\\&\qquad\leq\epsilon_{6}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}+\frac{4\delta^{2}\,(1-2\theta)^{2}\,\Delta t^{2}M^{2}\acute{d}^{2}}{\epsilon_{6}}\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\end{aligned}\tag{A.40}$$

$$(1-2\theta)\,\Delta t\left(g_{a}(\boldsymbol{\Gamma}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{4\,(1-2\theta)^{2}\,\Delta t^{2}M^{2}\acute{d}^{2}}{\epsilon_{7}}\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}+\epsilon_{7}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.41}$$

$$\begin{aligned}(1-2\theta)\,\Delta t\left(g_{a}(\boldsymbol{\Gamma}^{(\tilde{\theta})},\nabla\boldsymbol{u}^{(\tilde{\theta})}),\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right)\quad&\leq\quad\frac{4\delta\,(1-2\theta)\,\Delta tM^{2}\acute{d}^{2}}{\epsilon_{8}}\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^{2}\\&+\,\delta\,(1-2\theta)\,\Delta t\epsilon_{8}\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\end{aligned}\tag{A.42}$$

$$\left(\boldsymbol{F}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{1}{4\epsilon_{9}}\left\|\boldsymbol{F}^{(\theta)}\right\|^{2}+\epsilon_{9}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.43}$$

$$\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{1}{4\epsilon_{10}}\left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^{2}+\epsilon_{10}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.44}$$

$$\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{\omega^{2}\,(1-2\theta)^{2}\,\Delta t^{2}}{4\lambda^{2}\epsilon_{11}}\left\|\boldsymbol{F}^{(\theta)}\right\|^{2}+\epsilon_{11}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.45}$$

$$\begin{aligned}\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{F}^{(\theta)},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right)\quad&\leq\quad\frac{\delta\omega^{2}\,(1-2\theta)\,\Delta t}{\lambda^{2}4\epsilon_{12}}\left\|\boldsymbol{F}^{(\theta)}\right\|^{2}\\&+\,\delta\,(1-2\theta)\,\Delta t\epsilon_{12}\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\end{aligned}\tag{A.46}$$

$$\frac{\omega\,(1-2\theta)\,\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{\omega^{2}\,(1-2\theta)^{2}\,\Delta t^{2}}{\lambda^{2}4\epsilon_{13}}\left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^{2}+\epsilon_{13}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2},\tag{A.47}$$

$$\frac{\omega\left(1-2\theta\right)\Delta t}{\lambda}\left(\boldsymbol{\Gamma}^{(\theta)},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \leq \frac{\delta\omega^{2}\left(1-2\theta\right)\Delta t}{\lambda^{2}4\epsilon_{14}}\left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^{2} +\epsilon_{14}\delta\left(1-2\theta\right)\Delta t\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}, \tag{A.48}$$

$$\left(1-2\theta\right)\Delta t\left(\frac{\boldsymbol{\sigma}^{(\tilde{\theta})}-\boldsymbol{\sigma}^{(\theta)}}{\left(1-2\theta\right)\Delta t}-\boldsymbol{\sigma}_{t}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right) \leq \frac{\left(1-2\theta\right)^{2}\Delta t^{2}}{4\epsilon_{15}}\left\|\frac{\boldsymbol{\sigma}^{(\tilde{\theta})}-\boldsymbol{\sigma}^{(\theta)}}{\left(1-2\theta\right)\Delta t}-\boldsymbol{\sigma}_{t}^{(\tilde{\theta})}\right\|^{2} +\epsilon_{15}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}. \tag{A.49}$$

Using $\boldsymbol{u}\big|_{\partial\Omega}=0$, and $\nabla\cdot\boldsymbol{u}=0$,

$$\begin{aligned}\left(1-2\theta\right)\Delta t\left(\boldsymbol{\sigma}_{t}^{(\tilde{\theta})},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) &= -\left(1-2\theta\right)\delta\Delta t\left(\nabla\cdot\boldsymbol{u}^{(\tilde{\theta})},\boldsymbol{\sigma}_{t}^{(\tilde{\theta})}:\boldsymbol{F}^{(\tilde{\theta})}\right)\\ &\quad-\left(1-2\theta\right)\delta\Delta t\left(\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_{t}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)\\ &= -\left(1-2\theta\right)\delta\Delta t\left(\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_{t}^{(\tilde{\theta})},\boldsymbol{F}^{(\tilde{\theta})}\right)\\ &\leq \frac{\left(1-2\theta\right)^{2}\delta^{2}\Delta t^{2}}{4\epsilon_{16}}\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_{t}^{(\tilde{\theta})}\right\|^{2}+\epsilon_{16}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}.\end{aligned} \tag{A.50}$$

Lastly

$$\omega\left(1-2\theta\right)\Delta t\left(\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta},\boldsymbol{F}^{(\tilde{\theta})}\right)\leq\frac{\omega^{2}\left(1-2\theta\right)^{2}\Delta t^{2}}{4\epsilon_{17}}\left\|\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta}\right\|^{2}+\epsilon_{17}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}, \tag{A.51}$$

and

$$\omega\left(1-2\theta\right)\Delta t\left(\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta},\delta\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right) \leq \frac{\delta^{2}\omega^{2}\left(1-2\theta\right)^{2}\Delta t^{2}\acute{d}M^{2}Ch^{-2}}{4\epsilon_{18}}\left\|\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta}\right\|^{2} +\epsilon_{18}\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^{2}. \tag{A.52}$$

Using the bounds stated in (A.32) - (A.52) with (A.31) yields

$$
\begin{aligned}
&\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2\left(1+\frac{\tilde{\omega}\,(1-2\theta)\,\Delta t}{\lambda}-\epsilon_1-\epsilon_2-\epsilon_4-\epsilon_6-\epsilon_7-\epsilon_9\right.\\
&\qquad\left.-\epsilon_{10}-\epsilon_{11}-\epsilon_{13}-\epsilon_{15}-\epsilon_{16}-eps17-eps18-(1-2\theta)\,\Delta t 4M\acute{d}\right)\\
&\quad+\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2\delta\,(1-2\theta)\,\Delta t\left(1-\epsilon_3-\epsilon_5-\epsilon_8-\epsilon_{12}-\epsilon_{14}\right.\\
&\qquad\left.-\frac{4\delta\,(1-2\theta)\,\Delta t M^2\acute{d}^2}{\epsilon_6}\right)\\
\leq\;&\left\|\boldsymbol{F}^{(\theta)}\right\|^2\left(\frac{1}{4\epsilon_9}+\frac{\omega^2\,(1-2\theta)^2\,\Delta t^2}{4\lambda^2\epsilon_{11}}+\frac{\delta\omega^2\,(1-2\theta)\,\Delta t}{\lambda^2 4\epsilon_{12}}\right)\\
&+\left\|\boldsymbol{\Gamma}^{(\theta)}\right\|^2\left(\frac{1}{4\epsilon_{10}}+\frac{\omega^2\,(1-2\theta)^2\,\Delta t^2}{\lambda^2 4\epsilon_{13}}+\frac{\delta\omega^2\,(1-2\theta)\,\Delta t}{\lambda^2 4\epsilon_{14}}\right)\\
&+\left\|\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^2\left(\frac{1}{4\epsilon_1}+\frac{\tilde{\omega}^2\,(1-2\theta)^2\,\Delta t^2}{\lambda^2 4\epsilon_2}+\frac{\delta\tilde{\omega}^2\,(1-2\theta)\,\Delta t}{\lambda^2 4\epsilon_3}\right.\\
&\qquad\left.+\frac{4\,(1-2\theta)^2\,\Delta t^2M^2\acute{d}^2}{\epsilon_7}+\frac{4\delta\,(1-2\theta)\,\Delta t M^2\acute{d}^2}{\epsilon_8}\right)\\
&+\left\|\nabla\boldsymbol{\Gamma}^{(\tilde{\theta})}\right\|^2\left(\frac{(1-2\theta)^2\,\Delta t^2M^2\acute{d}}{4\epsilon_4}+\frac{\delta\,(1-2\theta)\,\Delta t M^2\acute{d}}{4\epsilon_5}\right)\\
&+\left\|\boldsymbol{\sigma}^{\tilde{\theta}}-\boldsymbol{\sigma}^{\theta}\right\|^2\left(\frac{\omega^2\,(1-2\theta)^2\,\Delta t^2}{4\epsilon_{17}}+\frac{\delta^2\omega^2\,(1-2\theta)^2\,\Delta t^2\acute{d}M^2Ch^{-2}}{4\epsilon_{18}}\right)\\
&+\frac{(1-2\theta)^2\,\Delta t^2}{4\epsilon_{15}}\left\|\frac{\boldsymbol{\sigma}^{(\tilde{\theta})}-\boldsymbol{\sigma}^{(\theta)}}{(1-2\theta)\,\Delta t}-\boldsymbol{\sigma}_t^{(\tilde{\theta})}\right\|^2+\frac{(1-2\theta)^2\,\delta^2\Delta t^2}{4\epsilon_{16}}\left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{\sigma}_t^{(\tilde{\theta})}\right\|^2. \qquad \text{(A.53)}
\end{aligned}
$$

Note that choosing $\epsilon_i (i \in \{1, \ldots, 18\})$ and Δt sufficiently small the L.H.S. of (A.53) will remain positive. Thus, using interpolation properties, and lemmas C.19, A.1, and C.20 we obtain

$$
\left\|\boldsymbol{F}^{(\tilde{\theta})}\right\|^2 \leq S(h,\Delta t,\delta) \qquad \text{and} \qquad \left\|\boldsymbol{u}^{(\tilde{\theta})}\cdot\nabla\boldsymbol{F}^{(\tilde{\theta})}\right\|^2 \leq S(h,\Delta t,\delta),
$$

where

$$
\begin{aligned}
S(h,\Delta t,\delta) \;=\;& C\big[h^{2m+2}+\Delta t^2h^{2m+2}+\delta^2\Delta t^2h^{2m}+\Delta t^2h^{2m}\\
&\qquad+\delta^2\Delta t^2h^{2m-2}+\Delta t^4+\delta^2\Delta t^2h^{-2}+\Delta t^4\delta^2h^{-2}\big]\left(1+\Delta t^2+\Delta t\delta\right)\\
&+C\left(h^{2m+2}+\Delta t^2h^{2m+2}+\Delta t\delta h^{2m+2}\right)\\
&+C\left(\Delta t^2h^{2m}+\Delta t\delta h^{2m}\right)+C(\Delta t)^4+C\delta^2\Delta t^2+C\delta^2\Delta t^4h^{-2}. \qquad \text{(A.54)}
\end{aligned}
$$

.

■

A.2 $\left\|\boldsymbol{E}^{(\theta)}\right\|^2$ and $\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2$

Lemma 5

$$\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + \frac{\theta 2(1-\alpha)\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|^2 \le Ch^{2k+2} + C(\Delta t)^4 + C\Delta t h^{2k}. \quad \text{(A.55)}$$

Proof: Subtracting ((4.3) with known $\boldsymbol{\sigma}$ and $n=0$) from ((3.3) at $t=\theta\Delta t$) yields:

$$\frac{Re}{\theta\Delta t}\left(\mathbf{e}_{\boldsymbol{u}}^{(\theta)}, \mathbf{v}\right) + 2(1-\alpha)\left(\mathbf{d}(\mathbf{e}_{\boldsymbol{u}}^{(\theta)}), \mathbf{d}(\mathbf{v})\right) = \frac{Re}{\theta\Delta t}\left(\mathbf{e}_{\boldsymbol{u}}^{(0)}, \mathbf{v}\right) + Re\left(\frac{\boldsymbol{u}^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta\Delta t} - \boldsymbol{u}_t^{(\theta)}, \mathbf{v}\right),\ \forall\, \mathbf{v} \in Z_h. \quad \text{(A.56)}$$

Recall that $\mathbf{e}_{\boldsymbol{u}}^{(\theta)} = \boldsymbol{\Lambda}^{(\theta)} + \boldsymbol{E}^{(\theta)}$. Let $\mathbf{v} = \boldsymbol{E}^{(\theta)}$ and from (A.56) we obtain

$$\begin{aligned}
&\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|^2 \\
&\quad = \left(\boldsymbol{u}^{(0)} - \boldsymbol{u}_h^{(0)}, \boldsymbol{E}^{(\theta)}\right) + \theta\Delta t\left(\frac{\boldsymbol{u}^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta\Delta t} - \boldsymbol{u}_t^{(\theta)}, \boldsymbol{E}^{(\theta)}\right) \\
&\qquad - \left(\boldsymbol{\Lambda}^{(\theta)}, \boldsymbol{E}^{(\theta)}\right) - \frac{2(1-\alpha)\theta\Delta t}{Re}\left(\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)}), \mathbf{d}(\boldsymbol{E}^{(\theta)})\right) \\
&\quad \le \frac{3}{2}\left\|\boldsymbol{u}^{(0)} - \boldsymbol{u}_h^{(0)}\right\|^2 + \frac{1}{6}\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + \frac{3\theta^2\Delta t^2}{2}\left\|\frac{\boldsymbol{u}^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta\Delta t} - \boldsymbol{u}_t^{(\theta)}\right\|^2 + \frac{1}{6}\left\|\boldsymbol{E}^{(\theta)}\right\|^2 \\
&\qquad + \frac{3}{2}\left\|\boldsymbol{\Lambda}^{(\theta)}\right\|^2 + \frac{1}{6}\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + \frac{(1-\alpha)\theta\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)})\right\|^2 + \frac{(1-\alpha)\theta\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|^2. \quad \text{(A.57)}
\end{aligned}$$

Simplifying expression (A.57) we obtain

$$\begin{aligned}
&\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|^2 \\
&\quad \le 3\left\|\boldsymbol{u}^{(0)} - \boldsymbol{u}_h^{(0)}\right\|^2 + 3\theta^2\Delta t^2\left\|\frac{\boldsymbol{u}^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta\Delta t} - \boldsymbol{u}_t^{(\theta)}\right\|^2 \\
&\qquad + 3\left\|\boldsymbol{\Lambda}^{(\theta)}\right\|^2 + \frac{2(1-\alpha)\theta\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)})\right\|^2. \quad \text{(A.58)}
\end{aligned}$$

Make note that

$$3\left\|\boldsymbol{u}^{(0)} - \boldsymbol{u}_h^{(0)}\right\|^2 = \left\|\boldsymbol{\Lambda}^{(0)}\right\|^2 \le Ch^{2k+2}\left\|\boldsymbol{u}(\cdot,0)\right\|_{k+1} \le Ch^{2k+2}. \quad \text{(A.59)}$$

Applying lemma C.18 and interpolation properties completes the proof.

■

Lemma A.3

$$\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2 \le Ch^{2k+2} + (1 + h^{-2}\Delta t)\left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta t h^{2k}\right] + C(\Delta t)^2 h^{2k-2}. \quad \text{(A.60)}$$

Proof: Subtracting ((4.4) with $n = 0$) from ((3.3) at $t = \theta\Delta t$):

$$\frac{Re}{(1-2\theta)\Delta t}\left(\mathbf{e}_{\boldsymbol{u}}^{(\tilde{\theta})}, \mathbf{v}\right) = \frac{Re}{(1-2\theta)\Delta t}\left(\mathbf{e}_{\boldsymbol{u}}^{(\theta)}, \mathbf{v}\right)$$
$$- 2(1-\alpha)\left(\mathbf{d}(\mathbf{e}_{\boldsymbol{u}}^{(\theta)}), \mathbf{d}(\mathbf{v})\right) + Re\left(\frac{\boldsymbol{u}^{(\tilde{\theta})} - \boldsymbol{u}^{(\theta)}}{(1-2\theta)\,\Delta t} - \boldsymbol{u}_t^{(\theta)}, \mathbf{v}\right) \forall\, \mathbf{v} \in Z_h. \quad \text{(A.61)}$$

Let $\mathbf{v} = \boldsymbol{E}^{(\tilde{\theta})}$ then (A.61) gives

$$\begin{aligned}
\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2 \;&=\; \left(\boldsymbol{E}^{(\theta)}, \boldsymbol{E}^{\tilde{\theta}}\right) + \left(\boldsymbol{\Lambda}^{(\theta)}, \boldsymbol{E}^{\tilde{\theta}}\right) - \left(\boldsymbol{\Lambda}^{(\tilde{\theta})}, \boldsymbol{E}^{\tilde{\theta}}\right) + (1-2\theta)\,\Delta t\left(\frac{\boldsymbol{u}^{(\tilde{\theta})} - \boldsymbol{u}^{(\theta)}}{(1-2\theta)\,\Delta t} - \boldsymbol{u}_t^{(\theta)}, \boldsymbol{E}^{\tilde{\theta}}\right) \\
&\quad - \frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left(\mathbf{d}(\boldsymbol{E}^{(\theta)}), \mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right) - \frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left(\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)}), \mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right) \\
&\leq\; \frac{4}{12}\left\|\boldsymbol{E}^{\tilde{\theta}}\right\|^2 + 3\left\|\boldsymbol{E}^{(\theta)}\right\|^2 + 3\left\|\boldsymbol{\Lambda}^{(\theta)}\right\|^2 + 3\left\|\boldsymbol{\Lambda}^{(\tilde{\theta})}\right\|^2 + 3\,(1-2\theta)\,\Delta t\left\|\frac{\boldsymbol{u}^{(\tilde{\theta})} - \boldsymbol{u}^{(\theta)}}{(1-2\theta)\,\Delta t} - \boldsymbol{u}_t^{(\theta)}\right\|^2 \\
&\quad + \frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|\left\|\mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right\| \\
&\quad + \frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)})\right\|\left\|\mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right\|. \qquad \text{(A.62)}
\end{aligned}$$

Note that
$$\left\|\mathbf{d}(\boldsymbol{E}^{(\tilde{\theta})}\right\| \leq \left\|\nabla\boldsymbol{E}^{(\tilde{\theta})}\right\| \leq Ch^{-1}\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|.$$

Considering the terms on the R.H.S. of expression (A.62) we have

$$\begin{aligned}
&\frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|\left\|\mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right\| \leq \frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\| Ch^{-1}\left\|\boldsymbol{E}^{\tilde{\theta}}\right\| \\
&\leq\; 3C^2h^{-2}\left(\frac{2(1-\alpha)\,(1-2\theta)^2\,\Delta t}{\theta Re}\right)\left(\frac{2(1-\alpha)\theta\Delta t}{Re}\right)\left\|\mathbf{d}(\boldsymbol{E}^{(\theta)})\right\|^2 + \frac{1}{12}\left\|\boldsymbol{E}^{\tilde{\theta}}\right\|^2 \\
&\leq\; 3C^2h^{-2}\left(\frac{2(1-\alpha)\,(1-2\theta)^2\,\Delta t}{\theta Re}\right)\left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta t h^{2k}\right] + \frac{1}{12}\left\|\boldsymbol{E}^{\tilde{\theta}}\right\|^2, \qquad \text{(A.63)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2(1-\alpha)\,(1-2\theta)\,\Delta t}{Re}\left\|\mathbf{d}(\boldsymbol{\Lambda}^{(\theta)})\right\|\left\|\mathbf{d}(\boldsymbol{E}^{\tilde{\theta}})\right\| \\
&\quad\leq\; 3C^2h^{-2}\left(\frac{4(1-\alpha)^2\,(1-2\theta)^2\,(\Delta t)^2}{Re^2}\right)Ch^{2k} + \frac{1}{12}\left\|\boldsymbol{E}^{\tilde{\theta}}\right\|^2. \qquad \text{(A.64)}
\end{aligned}$$

Applying lemma C.19, with the bounds established in (A.63), and (A.64) to (A.62) and we obtain:

$$\left\|\boldsymbol{E}^{(\tilde{\theta})}\right\|^2 \leq Ch^{2k+2} + (1 + h^{-2}\Delta t)\left[Ch^{2k+2} + C(\Delta t)^4 + C\Delta t h^{2k}\right] + C(\Delta t)^2 h^{2k-2}. \qquad \text{(A.65)}$$

■

B Taylor Expansions

The following are expressions for u at different times. Specifically the expressions listed are computations of Taylor series about the expansion points $u^{\left(n+\frac{1}{2}\right)}$, $u^{\left(n+\frac{1}{2}-\theta\right)}$, and $u^{\left(n+\theta-\frac{1}{2}\right)}$ that give an exact value for the truncation of the series. Integration by parts proves each of these statements.

Expanding about $u^{\left(n+\frac{1}{2}\right)}$:

$$\begin{aligned} u^{(n+1)} &= u^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_t(\cdot,t)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{tt}(\cdot,t)\left(t_{(n+1)}-t\right)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\frac{(\Delta t)^2}{4}u_{tt}^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{ttt}(\cdot,t)\left(t_{(n+1)}-t\right)^2\,dt \end{aligned} \tag{B.66}$$

$$\begin{aligned} u^{(n+\theta)} &= u^{\left(n+\frac{1}{2}\right)} - \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} - \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n+\theta)}\right)\,dt \end{aligned} \tag{B.67}$$

$$\begin{aligned} u^{(n+\tilde{\theta})} &= u^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} + \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{(n+\tilde{\theta})}-t\right)\,dt \end{aligned} \tag{B.68}$$

$$\begin{aligned} u^{(n)} &= u^{\left(n+\frac{1}{2}\right)} - \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} - \left(\frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right)\,dt \\ &= u^{\left(n+\frac{1}{2}\right)} - \left(\frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\frac{(\Delta t)^2}{4}u_{tt}^{\left(n+\frac{1}{2}\right)} \\ &\qquad -\frac{1}{2}\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{ttt}(\cdot,t)\left(t-t_{(n)}\right)^2\,dt \end{aligned} \tag{B.69}$$

Expanding about $u^{\left(n+\frac{1}{2}-\theta\right)}$:

$$\begin{aligned} u^{(n)} &= u^{\left(n+\frac{1}{2}-\theta\right)} - \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_t(\cdot,t)\,dt \\ &= u^{\left(n+\frac{1}{2}-\theta\right)} - \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right)\,dt \end{aligned} \tag{B.70}$$

$$
\begin{aligned}
u^{(n+\theta)} &= u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\frac{1}{2}-\theta\right)} + \left(2\theta-\frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)}-t\right)\,dt \qquad \text{(B.71)}
\end{aligned}
$$

$$
\begin{aligned}
u^{(n-\theta)} &= u^{\left(n+\frac{1}{2}-\theta\right)} - \int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\frac{1}{2}-\theta\right)} - \frac{\Delta t}{2} u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\,dt \qquad \text{(B.72)}
\end{aligned}
$$

$$
\begin{aligned}
u^{\left(n+\tilde{\theta}\right)} &= u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\frac{1}{2}-\theta\right)} + \frac{\Delta t}{2} u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)}-t\right)\,dt \qquad \text{(B.73)}
\end{aligned}
$$

Expanding about $u^{\left(n+\theta-\frac{1}{2}\right)}$:

$$
\begin{aligned}
u^{(n)} &= u^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\theta-\frac{1}{2}\right)} + \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)}-t\right)\,dt \qquad \text{(B.74)}
\end{aligned}
$$

$$
\begin{aligned}
u^{(n-\theta)} &= u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\theta-\frac{1}{2}\right)} - \left(2\theta-\frac{1}{2}\right)\Delta t u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\,dt \qquad \text{(B.75)}
\end{aligned}
$$

$$
\begin{aligned}
u^{(n+\theta-1)} &= u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\theta-\frac{1}{2}\right)} - \left(\frac{\Delta t}{2}\right) u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n+\theta-1)}\right)\,dt \qquad \text{(B.76)}
\end{aligned}
$$

$$
\begin{aligned}
u^{(n+\theta)} &= u^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt \\
&= u^{\left(n+\theta-\frac{1}{2}\right)} + \left(\frac{\Delta t}{2}\right) u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)}-t\right)\,dt \qquad \text{(B.77)}
\end{aligned}
$$

Expanding about $u^{\left(n-\frac{1}{2}\right)}$:

$$u^{(n)} = u^{\left(n-\frac{1}{2}\right)} + \frac{\Delta t}{2} u_t^{\left(n-\frac{1}{2}\right)} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{\left(n-\frac{1}{2}\right)} + \frac{1}{2} \int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot, t)(t_n - t)^2 \, dt \tag{B.78}$$

$$u^{(n-1)} = u^{\left(n-\frac{1}{2}\right)} - \frac{\Delta t}{2} u_t^{\left(n-\frac{1}{2}\right)} + \frac{1}{2} \frac{(\Delta t)^2}{4} u_{tt}^{\left(n-\frac{1}{2}\right)} - \frac{1}{2} \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot, t)(t - t_{n-1})^2 \, dt \tag{B.79}$$

C Lemmas

Lemma C.1

$$\left\|\theta u^{(n+1)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)}\right\|^2 \quad \leq \quad \frac{(\Delta t)^3}{24} \int_{t_{(n+\theta)}}^{t_{(n+1)}} \|u_{tt}(\cdot,t)\|^2 \; dt \tag{C.80}$$

Proof of Lemma C.1 using (B.66), (B.67) and choosing the value of $\theta = 1 - \frac{\sqrt{2}}{2}$:

$$\left\|\theta u^{(n+1)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)}\right\|^2 = \int_\Omega \left(\theta u^{(n+1)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)}\right)^2 dA$$

$$\begin{aligned}
= \quad & \int_\Omega \Bigg(\theta \left(u^{\left(n+\frac{1}{2}\right)} + \frac{1}{2}\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{tt}(\cdot,t)\left(t_{(n+1)} - t\right)\, dt\right) \\
& + (1-\theta)\left(u^{\left(n+\frac{1}{2}\right)} - \left(\frac{1}{2} - \theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta)}\right)\, dt\right) - u^{\left(n+\frac{1}{2}\right)}\Bigg)^2 dA \\
= \quad & \int_\Omega \Bigg(\left(-\theta^2 + 2\theta - \frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \theta \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{tt}(\cdot,t)\left(t_{(n+1)} - t\right)\, dt \\
& + (1-\theta)\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta)}\right)\, dt\Bigg)^2 dA \\
= \quad & \int_\Omega \left(\theta \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{tt}(\cdot,t)\left(t_{(n+1)} - t\right)\, dt + (1-\theta)\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta)}\right)\, dt\right)^2 dA \\
\leq \quad & 2\int_\Omega \left(\theta^2 \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} u_{tt}(\cdot,t)\left(t_{(n+1)} - t\right)\, dt\right)^2 + (1-\theta)^2 \left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta)}\right)\, dt\right)^2\right) dA \\
\leq \quad & 2\int_\Omega \Bigg(\theta^2 \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} \left(u_{tt}(\cdot,t)\right)^2\, dt \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} \left(t_{(n+1)} - t\right)^2\, dt\right) \\
& + (1-\theta)^2 \left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(t - t_{(n+\theta)}\right)^2\, dt\right)\Bigg) dA \\
= \quad & 2\int_\Omega \Bigg(\theta^2 \frac{1}{3}\left(\frac{\Delta t}{2}\right)^3 \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) \\
& + (1-\theta)^2\, \frac{1}{3}\left(\left(\frac{1}{2} - \theta\right)\Delta t\right)^3 \left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right)\Bigg) dA
\end{aligned}$$

$$= \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \int_\Omega \left(\theta^2 \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+1)}} \left(u_{tt}(\cdot,t)\right)^2\,dt\right) + (1-\theta)^2(1-2\theta)^3 \left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\,dt\right)\right) dA$$

$$\leq \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \max\left\{\theta^2, (1-\theta)^2, (1-2\theta)^3\right\} \int_\Omega \left(\int_{t_{(n+\theta)}}^{t_{(n+1)}} \left(u_{tt}(\cdot,t)\right)^2\,dt\right) dA$$

$$= \frac{(\Delta t)^3}{12}(1-\theta)^2 \int_{t_{(n+\theta)}}^{t_{(n+1)}} \left\|u_{tt}(\cdot,t)\right\|^2\,dt \qquad \blacksquare$$

Lemma C.2

$$\left\|\theta u^{(n)} + (1-\theta)\,u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}\right)}\right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n)}}^{t_{(n+\tilde{\theta})}} \left\|u_{tt}(\cdot,t)\right\|^2\,dt \qquad \text{(C.81)}$$

Proof of Lemma C.2 using (B.68), (B.69), and the value of $\theta = 1 - \frac{\sqrt{2}}{2}$:

$$\left\|\theta u^{(n)} + (1-\theta)\,u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}\right)}\right\|^2 = \int_\Omega \left(\theta u^{(n)} + (1-\theta)\,u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}\right)}\right)^2 dA$$

$$\begin{aligned}
&= \int_\Omega \Bigg(\theta\left(u^{\left(n+\frac{1}{2}\right)} - \left(\frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right)\,dt\right) \\
&\quad + (1-\theta)\left(u^{\left(n+\frac{1}{2}\right)} + \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)} - t\right)\,dt\right) - u^{\left(n+\frac{1}{2}\right)}\Bigg)^2 dA \\
&= \int_\Omega \Bigg(\left(\theta^2 - 2\theta + \frac{1}{2}\right)\Delta t\, u_t^{\left(n+\frac{1}{2}\right)} + \theta \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right)\,dt \\
&\quad + (1-\theta)\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)} - t\right)\,dt\Bigg)^2 dA \\
&= \int_\Omega \left(\theta \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right)\,dt + (1-\theta)\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)} - t\right)\,dt\right)^2 dA \\
&\leq 2\int_\Omega \left(\theta^2\left(\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right)\,dt\right)^2 + (1-\theta)^2\left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)} - t\right)\,dt\right)^2\right) dA \\
&\leq 2\int_\Omega \Bigg(\theta^2 \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\,dt \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(t - t_{(n)}\right)^2\,dt \\
&\quad + (1-\theta)^2 \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} \left(u_{tt}(\cdot,t)\right)^2\,dt \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} \left(t_{\left(n+\tilde{\theta}\right)} - t\right)^2\,dt\Bigg) dA \\
&= \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \int_\Omega \left(\theta^2 \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\,dt + (1-\theta)^2(1-2\theta)^3 \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} \left(u_{tt}(\cdot,t)\right)^2\,dt\right) dA
\end{aligned}$$

$$\leq \quad \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \max\left\{\theta^2, (1-\theta)^2, (1-2\theta)^3\right\} \int_\Omega \left(\int_{t_{(n)}}^{t_{(n+\tilde{\theta})}} \left(u_{tt}(\cdot,t)\right)^2 \, dt \right) dA$$

$$= \quad \frac{(\Delta t)^3}{12} (1-\theta)^2 \int_{t_{(n)}}^{t_{(n+\tilde{\theta})}} \|u_{tt}(\cdot,t)\|^2 \, dt \qquad \blacksquare$$

Lemma C.3

$$\left\|\theta u^{(n)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \leq \frac{(\Delta t)^3}{24} \int_{t_{(n)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot,t)\|^2 \, dt \tag{C.82}$$

Proof of Lemma C.3 using (B.70),(B.71) and choosing $\theta = \left(1 - \frac{\sqrt{2}}{2}\right)$:

$$\left\|\theta u^{(n)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 = \int_\Omega \left(\theta u^{(n)} + (1-\theta)\, u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right)^2 \, dA$$

$$\begin{aligned}
= \quad & \int_\Omega \Bigg(\theta \left(u^{\left(n+\frac{1}{2}-\theta\right)} - \left(\frac{1}{2}-\theta\right) \Delta t \, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right) \, dt \right) \\
& + (1-\theta) \left(u^{\left(n+\frac{1}{2}-\theta\right)} + \left(2\theta - \frac{1}{2}\right) \Delta t \, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right) \, dt \right) - u^{\left(n+\frac{1}{2}-\theta\right)} \Bigg)^2 dA \\
= \quad & \int_\Omega \Bigg(\left(-\theta^2 + 2\theta - \frac{1}{2}\right) \Delta t \, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \theta \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right) \, dt \\
& + (1-\theta) \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right) \, dt \Bigg)^2 dA \\
= \quad & \int_\Omega \left(\theta \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right) \, dt + (1-\theta) \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right) \, dt \right)^2 dA \\
\leq \quad & 2 \int_\Omega \left(\theta^2 \left(\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t - t_{(n)}\right) \, dt \right)^2 + (1-\theta)^2 \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right) \, dt \right)^2 \right) dA \\
\leq \quad & 2 \int_\Omega \Bigg(\theta^2 \left(\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} \left(u_{tt}(\cdot,t)\right)^2 \, dt \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} \left(t - t_{(n)}\right)^2 \, dt \right) \\
& + (1-\theta)^2 \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2 \, dt \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(t_{(n+\theta)} - t\right)^2 \, dt \right) \Bigg) dA
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}\int_\Omega \left(\theta^2\left(\left(\frac{1}{2}-\theta\right)\Delta t\right)^3 \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right.\\
&\quad \left. + (1-\theta)^2\left(\left(2\theta-\frac{1}{2}\right)\Delta t\right)^3 \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA\\
&= \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \int_\Omega \left(\theta^2(1-2\theta)^3 \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt + (1-\theta)^2(4\theta-1)^3 \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA\\
&\le \frac{(\Delta t)^3}{12}\max\left\{\theta^2,(1-2\theta)^3,(1-\theta)^2,(4\theta-1)^3\right\}\int_\Omega \left(\int_{t_{(n)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA\\
&= \frac{(\Delta t)^3}{12}(1-\theta)^2 \int_\Omega \left(\int_{t_{(n)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA\\
&= \frac{(\Delta t)^3}{12}(1-\theta)^2 \int_{t_{(n)}}^{t_{(n+\theta)}} \left\|u_{tt}(\cdot,t)\right\|^2\, dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.4

$$
\left\|(1-2\theta)\,u^{\left(n+\tilde{\theta}\right)} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \le \frac{(\Delta t)^3}{3}\left(3-2\sqrt{2}\right)\int_{t_{(n-\theta)}}^{t_{(n+\tilde{\theta})}} \left\|u_{tt}(\cdot,t)\right\|^2\, dt \tag{C.83}
$$

Proof of Lemma C.4 using (B.70), (B.72),(B.73) and choosing $\theta = \left(1-\frac{\sqrt{2}}{2}\right)$:

$$
\left\|(1-2\theta)\,u^{\left(n+\tilde{\theta}\right)} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2
$$

$$
\begin{aligned}
&= \int_\Omega \left((1-2\theta)\,u^{\left(n+\tilde{\theta}\right)} + \theta u^{(n)} + \theta u^{(n-\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)}\right)^2 dA\\
&= \int_\Omega \left((1-2\theta)\left(u^{\left(n+\frac{1}{2}-\theta\right)} + \frac{\Delta t}{2}u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)}-t\right) dt\right)\right.\\
&\quad +\theta\left(u^{\left(n+\frac{1}{2}-\theta\right)} - \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right) dt\right)\\
&\quad \left. +\theta\left(u^{\left(n+\frac{1}{2}-\theta\right)} - \frac{\Delta t}{2}u_t^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right) dt\right) - u^{\left(n+\frac{1}{2}-\theta\right)}\right)^2 dA\\
&= \int_\Omega \left(\left((1-2\theta)-\theta(1-2\theta)-\theta\right)\frac{\Delta t}{2}u_t^{\left(n+\frac{1}{2}-\theta\right)} + (1-2\theta)\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)}-t\right) dt\right.\\
&\quad \left. +\theta\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right) dt + \theta\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right) dt\right)^2 dA\\
&= \int_\Omega \left((1-2\theta)\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)}-t\right) dt + \theta\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right) dt\right.\\
&\quad \left. +\theta\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right) dt\right)^2 dA
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 4\int_{\Omega}\left((1-2\theta)^2\left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_{tt}(\cdot,t)\left(t_{\left(n+\tilde{\theta}\right)}-t\right)\,dt\right)^2+\theta^2\left(\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n)}\right)\,dt\right)^2\right.\\
& \left.+\theta^2\left(\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\,dt\right)^2\right)dA\\
\leq\ & 4\int_{\Omega}\left((1-2\theta)^2\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}}\left(u_{tt}(\cdot,t)\right)^2\,dt\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}}\left(t_{\left(n+\tilde{\theta}\right)}-t\right)^2\,dt\right.\\
& +\theta^2\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(t-t_{(n)}\right)^2\,dt\\
& \left.+\theta^2\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(t-t_{(n-\theta)}\right)^2\,dt\right)dA
\end{aligned}
$$

$$
\begin{aligned}
=\ & \frac{4}{3}\int_{\Omega}\left((1-2\theta)^2\left(\frac{\Delta t}{2}\right)^3\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}}\left(u_{tt}(\cdot,t)\right)^2\,dt+\theta^2\left((1-2\theta)\frac{\Delta t}{2}\right)^3\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\right.\\
& \left.+\theta^2\left(\frac{\Delta t}{2}\right)^3\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\right)dA\\
=\ & \frac{4}{3}\left(\frac{\Delta t}{2}\right)^3\int_{\Omega}\left((1-2\theta)^2\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}}\left(u_{tt}(\cdot,t)\right)^2\,dt+\theta^2\,(1-2\theta)^3\int_{t_{(n)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\right.\\
& \left.+\theta^2\int_{t_{(n-\theta)}}^{t_{\left(n+\frac{1}{2}-\theta\right)}}\left(u_{tt}(\cdot,t)\right)^2\,dt\right)dA\\
\leq\ & \frac{4}{3}\left(\frac{\Delta t}{2}\right)^3 2\max\left\{(1-2\theta)^2,\theta^2,(1-2\theta)^3\right\}\int_{\Omega}\left(\int_{t_{(n-\theta)}}^{t_{(n+\tilde{\theta})}}\left(u_{tt}(\cdot,t)\right)^2\,dt\right)dA\\
=\ & \frac{(\Delta t)^3}{3}\,(1-2\theta)^2\int_{t_{(n-\theta)}}^{t_{(n+\tilde{\theta})}}\left\|u_{tt}(\cdot,t)\right\|^2\,dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.5

$$
\left\|\theta u^{(n)}+(1-\theta)\,u^{(n-\theta)}-u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2\leq\frac{(\Delta t)^3}{24}\int_{t_{(n-\theta)}}^{t_{(n)}}\left\|u_{tt}(\cdot,t)\right\|^2\,dt
$$

Proof of Lemma C.5 using (B.74), (B.75) and the value of $\theta=\left(1-\frac{\sqrt{2}}{2}\right)$.

$$
\left\|\theta u^{(n)}+(1-\theta)\,u^{(n-\theta)}-u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2=\int_{\Omega}\left(\theta u^{(n)}+(1-\theta)\,u^{(n-\theta)}-u^{\left(n+\theta-\frac{1}{2}\right)}\right)^2\,dA
$$

$$
\begin{aligned}
&= \int_\Omega \left(\theta u^{(n)} + (1-\theta)\, u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right)^2 dA \\
&= \int_\Omega \left(\theta\left(u^{\left(n+\theta-\frac{1}{2}\right)} + \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)}-t\right)\, dt\right)\right. \\
&\quad \left. +(1-\theta)\left(u^{\left(n+\theta-\frac{1}{2}\right)} - \left(2\theta-\frac{1}{2}\right)\Delta t u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\, dt\right) - u^{\left(n+\theta-\frac{1}{2}\right)}\right)^2 dA \\
&= \int_\Omega \left(\left(\theta^2-2\theta+\frac{1}{2}\right)\Delta t u_t^{\left(n+\theta-\frac{1}{2}\right)} + \theta\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)}-t\right)\, dt\right. \\
&\quad \left. +(1-\theta)\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\, dt\right)^2 dA \\
&= \int_\Omega \left(\theta\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)}-t\right)\, dt + (1-\theta)\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\, dt\right)^2 dA \\
&\le 2\int_\Omega \left(\theta^2\left(\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)}-t\right)\, dt\right)^2 + (1-\theta)^2\left(\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t-t_{(n-\theta)}\right)\, dt\right)^2\right) dA \\
&\le 2\int_\Omega \left(\theta^2\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(u_{tt}(\cdot,t)\right)^2\, dt \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(t_{(n)}-t\right)^2\, dt\right. \\
&\quad \left. +(1-\theta)^2\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(t-t_{(n-\theta)}\right)^2\, dt\right) dA \\
&= \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \int_\Omega \left(\theta^2(1-2\theta)^3\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(u_{tt}(\cdot,t)\right)^2\, dt + (1-\theta)^2(4\theta-1)^3\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA \\
&\le \frac{2}{3}\left(\frac{\Delta t}{2}\right)^3 \max\left\{\theta^2, (1-2\theta)^3, (1-\theta)^2, (4\theta-1)^3\right\} \int_\Omega \left(\int_{t_{(n-\theta)}}^{t_{(n)}} \left(u_{tt}(\cdot,t)\right)^2\, dt\right) dA \\
&= \frac{(\Delta t)^3}{12}(1-\theta)^2 \int_{t_{(n-\theta)}}^{t_{(n)}} \|u_{tt}(\cdot,t)\|^2\, dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.6

$$
\left\|(1-2\theta)\, u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \le \frac{(\Delta t)^3}{3}\left(3-2\sqrt{2}\right)\int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} \|u_{tt}(\cdot,t)\|^2\, dt
$$

Proof of Lemma C.6 using (B.74),(B.76),(B.77) and the value of $\theta = \left(1-\frac{\sqrt{2}}{2}\right)$:

$$
\left\|(1-2\theta)\, u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 = \int_\Omega \left((1-2\theta)\, u^{(n+\theta-1)} + \theta u^{(n)} + \theta u^{(n+\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right)^2 dA
$$

$$
\begin{aligned}
&= \int_\Omega \Bigg((1-2\theta)\left(u^{\left(n+\theta-\frac{1}{2}\right)} - \left(\frac{\Delta t}{2}\right) u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta-1)}\right)\, dt \right) \\
&\quad +\theta \left(u^{\left(n+\theta-\frac{1}{2}\right)} + \left(\frac{1}{2}-\theta\right)\Delta t\, u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)} - t\right)\, dt \right) \\
&\quad +\theta \left(u^{\left(n+\theta-\frac{1}{2}\right)} + \left(\frac{\Delta t}{2}\right) u_t^{\left(n+\theta-\frac{1}{2}\right)} + \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right)\, dt \right) - u^{\left(n+\theta-\frac{1}{2}\right)} \Bigg)^2 dA \\
&= \int_\Omega \Bigg(\left(-2\theta^2 + 4\theta - 1\right)\left(\frac{\Delta t}{2}\right) u_t^{\left(n+\theta-\frac{1}{2}\right)} + (1-2\theta)\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta-1)}\right)\, dt \\
&\quad +\theta \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)} - t\right)\, dt + \theta \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right)\, dt \Bigg)^2 dA \\
&= \int_\Omega \Bigg((1-2\theta)\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta-1)}\right)\, dt + \theta \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)} - t\right)\, dt \\
&\quad +\theta \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right)\, dt \Bigg)^2 dA \\
&\leq 4\int_\Omega \Bigg((1-2\theta)^2 \left(\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_{tt}(\cdot,t)\left(t - t_{(n+\theta-1)}\right)\, dt \right)^2 + \theta^2 \left(\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} u_{tt}(\cdot,t)\left(t_{(n)} - t\right)\, dt \right)^2 \\
&\quad +\theta^2 \left(\int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} u_{tt}(\cdot,t)\left(t_{(n+\theta)} - t\right)\, dt \right)^2 \Bigg) dA \\
&\leq 4\int_\Omega \Bigg((1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)\, dt \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(t - t_{(n+\theta-1)}\right)^2 dt \\
&\quad +\theta^2 \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(u_{tt}(\cdot,t)\right)^2 dt \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(t_{(n)} - t\right)^2 dt \\
&\quad +\theta^2 \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2 dt \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} \left(t_{(n+\theta)} - t\right)^2 dt \Bigg) dA \\
&\leq \frac{4}{3}\left(\frac{\Delta t}{2}\right)^3 \int_\Omega \Bigg((1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_{tt}(\cdot,t)\right)\, dt + \theta^2(1-2\theta)^3 \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n)}} \left(u_{tt}(\cdot,t)\right)^2 dt \\
&\quad +\theta^2 \int_{t_{\left(n+\theta-\frac{1}{2}\right)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2 dt \Bigg) dA \\
&\leq \frac{8}{3}\left(\frac{\Delta t}{2}\right)^3 \max\left\{ (1-2\theta)^2, \theta^2, (1-2\theta)^3 \right\} \int_\Omega \left(\int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} \left(u_{tt}(\cdot,t)\right)^2 dt \right) dA \\
&= \frac{(\Delta t)^3}{3}(1-2\theta)^2 \int_{t_{(n+\theta-1)}}^{t_{(n+\theta)}} \left\|u_{tt}(\cdot,t)\right\|^2 dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.7

$$\left\| u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)} \right\|^2 \leq \Delta t \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \|u_t(\cdot,t)\|^2 \; dt$$

Proof of Lemma C.7 using (B.67):

$$\begin{aligned}
\left\| u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)} \right\|^2 &= \int_\Omega \left(u^{(n+\theta)} - u^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(u^{\left(n+\frac{1}{2}\right)} - \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\, dt - u^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(- \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} 1\, u_t(\cdot,t)\, dt \right)^2 dA \\
&\leq \int_\Omega \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} 1^2\, dt \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} (u_t(\cdot,t))^2 \; dt\, dA \\
&\leq \int_\Omega \int_{t_{(n)}}^{t_{(n+1)}} 1^2\, dt \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} (u_t(\cdot,t))^2 \; dt\, dA \\
&\leq \Delta t \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} \|u_t(\cdot,t)\|^2 \; dt \qquad \blacksquare
\end{aligned}$$

Lemma C.8

$$\left\| u^{(n+\tilde{\theta})} - u^{\left(n+\frac{1}{2}\right)} \right\|^2 \leq \Delta t \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} \|u_t(\cdot,t)\|^2 \; dt$$

Proof of Lemma C.8 using (B.68):

$$\begin{aligned}
\left\| u^{(n+\tilde{\theta})} - u^{\left(n+\frac{1}{2}\right)} \right\|^2 &= \int_\Omega \left(u^{(n+\tilde{\theta})} - u^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(u^{\left(n+\frac{1}{2}\right)} + \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\, dt - u^{\left(n+\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} 1\, u_t(\cdot,t)\, dt \right)^2 dA \\
&\leq \int_\Omega \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} 1^2\, dt \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} (u_t(\cdot,t))^2 \; dt\, dA \\
&\leq \int_\Omega \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\frac{1}{2}+1\right)}} 1^2\, dt \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} (u_t(\cdot,t))^2 \; dt\, dA \\
&\leq \Delta t \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{(n+\tilde{\theta})}} \|u_t(\cdot,t)\|^2 \; dt \qquad \blacksquare
\end{aligned}$$

Lemma C.9

$$\left\|\frac{1}{2}u^{(n+\theta)}+\frac{1}{2}u^{\left(n+\tilde{\theta}\right)}-u^{\left(n+\frac{1}{2}\right)}\right\|^2 \le \frac{\Delta t}{2}\int_{t_{(n+\theta)}}^{t_{\left(n+\tilde{\theta}\right)}} \|u_t(\cdot,t)\|^2\, dt$$

Proof of Lemma C.9 using (B.67) and (B.68):

$$\begin{aligned}
\left\|\frac{1}{2}u^{(n+\theta)}+\frac{1}{2}u^{\left(n+\tilde{\theta}\right)}-u^{\left(n+\frac{1}{2}\right)}\right\|^2 &= \int_\Omega \left(\frac{1}{2}u^{(n+\theta)}+\frac{1}{2}u^{\left(n+\tilde{\theta}\right)}-u^{\left(n+\frac{1}{2}\right)}\right)^2 dA \\
&= \int_\Omega \left(\frac{1}{2}\left(u^{\left(n+\frac{1}{2}\right)}-\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\,dt\right)\right. \\
&\quad \left. +\frac{1}{2}\left(u^{\left(n+\frac{1}{2}\right)}+\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} u_t(\cdot,t)\,dt\right)-u^{\left(n+\frac{1}{2}\right)}\right)^2 dA \\
&= \frac{1}{4}\int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} u_t(\cdot,t)\,dt-\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\,dt\right)^2 dA \\
&\le \frac{1}{2}\int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} u_t(\cdot,t)\,dt\right)^2+\left(\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} u_t(\cdot,t)\,dt\right)^2 dA \\
&\le \frac{1}{2}\int_\Omega \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} 1\,dt \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} (u_t(\cdot,t))^2\, dt \\
&\quad +\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} 1\,dt \int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} (u_t(\cdot,t))^2\, dt\,dA \\
&\le \frac{\Delta t}{2}\int_\Omega \int_{t_{\left(n+\frac{1}{2}\right)}}^{t_{\left(n+\tilde{\theta}\right)}} (u_t(\cdot,t))^2\, dt+\int_{t_{(n+\theta)}}^{t_{\left(n+\frac{1}{2}\right)}} (u_t(\cdot,t))^2\, dt\,dA \\
&\le \frac{\Delta t}{2}\int_{t_{(n+\theta)}}^{t_{\left(n+\tilde{\theta}\right)}} \|u_t(\cdot,t)\|^2\, dt
\end{aligned}$$

■

Lemma C.10

$$\left\|u^{(n+\theta)}-u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 \le \Delta t\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \|u_t(\cdot,t)\|^2\, dt$$

Proof of Lemma C.10 using (B.71):

$$\left\|u^{(n+\theta)}-u^{\left(n+\frac{1}{2}-\theta\right)}\right\|^2 = \int_\Omega \left(u^{(n+\theta)}-u^{\left(n+\frac{1}{2}-\theta\right)}\right)^2 dA$$

$$
\begin{aligned}
&= \int_\Omega \left(u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt - u^{\left(n+\frac{1}{2}-\theta\right)} \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} 1\, u_t(\cdot,t)\,dt \right)^2 dA \\
&\leq \int_\Omega \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} 1^2\,dt \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_t(\cdot,t)\right)^2\,dt\,dA \\
&\leq \Delta t \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left\|u_t(\cdot,t)\right\|^2\,dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.11

$$
\left\| u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 \leq \Delta t \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left\|u_t(\cdot,t)\right\|^2\,dt
$$

Proof of Lemma C.11 using (B.73):

$$
\begin{aligned}
\left\| u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 &= \int_\Omega \left(u^{\left(n+\tilde{\theta}\right)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right)^2 dA \\
&= \int_\Omega \left(u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt - u^{\left(n+\frac{1}{2}-\theta\right)} \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} 1\, u_t(\cdot,t)\,dt \right)^2 dA \\
&\leq \int_\Omega \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} 1^2\,dt \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left(u_t(\cdot,t)\right)^2\,dt\,dA \\
&\leq \Delta t \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left\|u_t(\cdot,t)\right\|^2\,dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.12

$$
\left\| \frac{1}{2} u^{\left(n+\tilde{\theta}\right)} + \frac{1}{2} u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 \leq \Delta t \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left\|u_t(\cdot,t)\right\|^2\,dt
$$

Proof of Lemma C.12 using (B.71), and (B.73):

$$
\begin{aligned}
\left\| \frac{1}{2} u^{\left(n+\tilde{\theta}\right)} + \frac{1}{2} u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right\|^2 &= \int_\Omega \left(\frac{1}{2} u^{\left(n+\tilde{\theta}\right)} + \frac{1}{2} u^{(n+\theta)} - u^{\left(n+\frac{1}{2}-\theta\right)} \right)^2 dA \\
&= \int_\Omega \left(\frac{1}{2} \left(u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt \right) \right. \\
&\qquad \left. + \frac{1}{2} \left(u^{\left(n+\frac{1}{2}-\theta\right)} + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt \right) - u^{\left(n+\frac{1}{2}-\theta\right)} \right)^2 dA
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\int_\Omega \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt \right)^2 dA \\
&\leq \frac{1}{2}\int_\Omega \left(\left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} u_t(\cdot,t)\,dt \right)^2 + \left(\int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} u_t(\cdot,t)\,dt \right)^2 \right) dA \\
&\leq \frac{1}{2}\int_\Omega \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} 1^2\,dt \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left(u_t(\cdot,t)\right)^2\,dt + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} 1^2\,dt \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_t(\cdot,t)\right)^2\,dt\,dA \\
&\leq \frac{\Delta t}{2}\int_\Omega \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left(u_t(\cdot,t)\right)^2\,dt + \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\theta)}} \left(u_t(\cdot,t)\right)^2\,dt\,dA \\
&\leq \Delta t \int_{t_{\left(n+\frac{1}{2}-\theta\right)}}^{t_{(n+\tilde{\theta})}} \left\|u_t(\cdot,t)\right\|^2\,dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.13

$$
\left\|u^{(n+\theta-1)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left\|u_t(\cdot,t)\right\|^2\,dt
$$

Proof of Lemma C.13 using (B.76):

$$
\begin{aligned}
\left\|u^{(n+\theta-1)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 &= \int_\Omega \left(u^{(n+\theta-1)} - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \right)^2 dA \\
&\leq \int_\Omega \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} 1^2\,dt \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\,dt\,dA \\
&\leq \Delta t \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left\|u_t(\cdot,t)\right\|^2\,dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.14

$$
\left\|u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 \leq \Delta t \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left\|u_t(\cdot,t)\right\|^2\,dt
$$

Proof of Lemma C.14 using (B.75):

$$
\begin{aligned}
\left\|u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)}\right\|^2 &= \int_\Omega \left(u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\,dt \right)^2 dA
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_\Omega \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} 1^2\, dt \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\, dt\, dA \\
&\le \Delta t \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \|u_t(\cdot,t)\|^2\, dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.15

$$
\left\| \frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)} \right\|^2 \le \Delta t \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \|u_t(\cdot,t)\|^2\, dt
$$

Proof of Lemma C.15 using (B.75), and (B.76):

$$
\begin{aligned}
\left\| \frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)} \right\|^2 &= \int_\Omega \left(\frac{1}{2}u^{(n+\theta-1)} + \frac{1}{2}u^{(n-\theta)} - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \int_\Omega \left(\frac{1}{2}\left(u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right) \right. \\
&\quad \left. + \frac{1}{2}\left(u^{\left(n+\theta-\frac{1}{2}\right)} - \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right) - u^{\left(n+\theta-\frac{1}{2}\right)} \right)^2 dA \\
&= \frac{1}{4}\int_\Omega \left(\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt + \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right)^2 dA \\
&\le \frac{1}{2}\int_\Omega \left(\int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right)^2 + \left(\int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} u_t(\cdot,t)\, dt \right)^2 dA \\
&\le \frac{1}{2}\int_\Omega \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} 1^2\, dt \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\, dt \\
&\quad + \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} 1^2\, dt \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\, dt\, dA \\
&\le \frac{\Delta t}{2}\int_\Omega \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\, dt + \int_{t_{(n-\theta)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \left(u_t(\cdot,t)\right)^2\, dt\, dA \\
&\le \Delta t \int_{t_{(n+\theta-1)}}^{t_{\left(n+\theta-\frac{1}{2}\right)}} \|u_t(\cdot,t)\|^2\, dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.16

$$
\left\| \frac{u^{(n)} - u^{(n-1)}}{\Delta t} - u_t^{\left(n-\frac{1}{2}\right)} \right\|^2 \le \frac{\Delta t^3}{5\,(2^6)} \int_{t_{n-1}}^{t_n} \|u_{ttt}(\cdot,t)\|^2\, dt
$$

Proof of Lemma C.16 using (B.78) and (B.79):

$$
\left\| \frac{u^{(n)} - u^{(n-1)}}{\Delta t} - u_t^{\left(n-\frac{1}{2}\right)} \right\|^2 = \left\| \frac{1}{\Delta t}\left(u^{(n)} - u^{(n-1)} \right) - u_t^{\left(n+\frac{1}{2}\right)} \right\|^2
$$

$$
\begin{aligned}
= \int_\Omega & \left(\frac{1}{\Delta t}u^{\left(n-\frac{1}{2}\right)} + \frac{1}{2}u_t^{\left(n-\frac{1}{2}\right)} + \frac{1}{2}\frac{(\Delta t)}{4}u_{tt}^{\left(n-\frac{1}{2}\right)} + \frac{1}{2\Delta t}\int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot,t)(t_n - t)^2\, dt \right. \\
& \left. -\frac{1}{\Delta t}u^{\left(n-\frac{1}{2}\right)} + \frac{1}{2}u_t^{\left(n-\frac{1}{2}\right)} - \frac{1}{2}\frac{(\Delta t)}{4}u_{tt}^{\left(n-\frac{1}{2}\right)} + \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot,t)(t - t_{n-1})^2\, dt - u_t^{\left(n-\frac{1}{2}\right)} \right)^2 dA
\end{aligned}
$$

$$
\begin{aligned}
&= \int_\Omega \left(\frac{1}{2\Delta t} \int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot,t)(t_n - t)^2\,dt + \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot,t)(t - t_{n-1})^2\,dt \right)^2 dA \\
&\le \int_\Omega \left(\frac{1}{2\Delta t}\right)^2 \left(2 \left(\int_{t_{n-\frac{1}{2}}}^{t_n} u_{ttt}(\cdot,t)(t_n - t)^2\,dt \right)^2 + 2 \left(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} u_{ttt}(\cdot,t)(t - t_{n-1})^2\,dt \right)^2 \right) dA \\
&\le 2 \left(\frac{1}{2\Delta t}\right)^2 \int_\Omega \left(\int_{t_{n-\frac{1}{2}}}^{t_n} (u_{ttt}(\cdot,t))^2\ dt \int_{t_{n-\frac{1}{2}}}^{t_n} (t_n - t)^4\,dt \right. \\
&\quad \left. + \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (u_{ttt}(\cdot,t))^2\ dt \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (t - t_{n-1})^4\,dt \right) dA \\
&= 2 \left(\frac{1}{2\Delta t}\right)^2 \int_\Omega \left(\frac{1}{5} \left(\frac{\Delta t}{2}\right)^5 \left(\int_{t_{n-\frac{1}{2}}}^{t_n} (u_{ttt}(\cdot,t))^2\ dt + \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (u_{ttt}(\cdot,t))^2\ dt \right) \right) dA \\
&= \frac{\Delta t^3}{5\,(2^6)} \int_\Omega \int_{t_{n-1}}^{t_n} (u_{ttt}(\cdot,t))^2\ dt\,dA \\
&= \frac{\Delta t^3}{5\,(2^6)} \int_{t_{n-1}}^{t_n} \|u_{ttt}(\cdot,t)\|^2\ dt \qquad \blacksquare
\end{aligned}
$$

Lemma C.17 *Assuming* $\|u_{tt}(\cdot,t)\| < M$ *then*

$$
\left\| \frac{u^{(\theta)} - u^{(0)}}{\theta\Delta t} - u_t^{(0)} \right\|^2 \le C(\Delta t)^2.
$$

Proof of Lemma C.17: Make note that

$$
u^{(\theta)} - u^{(0)} = \int_0^{\theta\Delta t} u_t(\cdot,t)\,dt = (\theta\Delta t)\,u_t(\cdot,0) - \int_0^{\theta\Delta t} (t - \theta\Delta t)\,u_{tt}(\cdot,t)\,dt.
$$

$$
\begin{aligned}
\implies \left| \frac{u^{(\theta)} - u^{(0)}}{\theta\Delta t} - u_t^{(0)} \right| &= \frac{1}{\theta\Delta t} \left| \int_0^{\theta\Delta t} (\theta\Delta t - t) u_{tt}(\cdot,t)\,dt \right| \\
&\le \frac{1}{\theta\Delta t} \left(\int_0^{\theta\Delta t} (\theta\Delta t - t)^2\,dt \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot,t)|^2\ dt \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{3}} (\theta\Delta t)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot,t)|^2\ dt \right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\| \frac{u^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta\Delta t} - u_t^{(0)} \right\|^2 &\le \frac{1}{3}\theta\Delta t \int_\Omega \int_{t_0}^{t_\theta} |u_{tt}(\cdot,t)|^2\ dt\,dA \\
&= \frac{1}{3}\theta\Delta t \int_{t_0}^{t_\theta} \int_\Omega |u_{tt}(\cdot,t)|^2\ dA\,dt \\
&\le \frac{1}{3}(\theta\Delta t)^2 M^2 \\
&\le C(\Delta t)^2. \qquad \blacksquare
\end{aligned}
$$

Lemma C.18 *Assuming* $\|u_{tt}(\cdot,t)\| < M$ *then*

$$\left\| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.18: Make note that

$$u^{(\theta)} - u^{(0)} = \int_{t_0}^{t_\theta} u_t(\cdot,t)\,dt = (\theta\Delta t)\,u_t(\cdot,\theta\Delta t) - \int_{t_0}^{t_\theta} t\,u_{tt}(\cdot,t)\,dt.$$

$$\begin{aligned}
\implies \left| \frac{u^{(\theta)} - u^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right| &= \frac{1}{\theta\Delta t}\left| \int_0^{\theta\Delta t} t u_{tt}(\cdot,t)\,dt \right| \\
&\leq \frac{1}{\theta\Delta t}\left(\int_{t_0}^{t_\theta} t^2\,dt \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot,t)|^2\,dt \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{3}}(\theta\Delta t)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} |u_{tt}(\cdot,t)|^2\,dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| \frac{u^{(\theta)} - \boldsymbol{u}^{(0)}}{\theta \Delta t} - u_t^{(\theta)} \right\|^2 &\leq \frac{1}{3}\theta\Delta t \int_\Omega \int_0^{\theta\Delta t} |u_{tt}(\cdot,t)|^2\,dt\,dA \\
&= \frac{1}{3}\theta\Delta t \int_{t_0}^{t_\theta} \int_\Omega |u_{tt}(\cdot,t)|^2\,dA\,dt \\
&\leq \frac{1}{3}(\theta\Delta t)^2 M^2 \\
&\leq C(\Delta t)^2. \qquad \blacksquare
\end{aligned}$$

Lemma C.19 *Assuming* $\|u_{tt}(\cdot,t)\| < M$ *then*

$$\left\| \frac{u^{(\tilde\theta)} - u^{(\theta)}}{(1-2\theta)\,\Delta t} - u_t^{(\theta)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.19: Make note that

$$u^{(\tilde\theta)} - u^{(\theta)} = \int_{t_\theta}^{t_{\tilde\theta}} u_t(\cdot,t)\,dt = ((1-2\theta)\,\Delta t)\,u_t(\cdot,\theta\Delta t) - \int_{t_\theta}^{t_{\tilde\theta}} (t-\tilde\theta\Delta t) u_{tt}(\cdot,t)\,dt.$$

$$\begin{aligned}
\implies \left| \frac{u^{(\tilde\theta)} - u^{(\theta)}}{(1-2\theta)\,\Delta t} - u_t^{(\theta)} \right| &= \frac{1}{(1-2\theta)\,\Delta t}\left| \int_{t_\theta}^{t_{\tilde\theta}} (\tilde\theta\Delta t - t) u_{tt}(\cdot,t)\,dt \right| \\
&\leq \frac{1}{(1-2\theta)\,\Delta t}\left(\int_{t_\theta}^{t_{\tilde\theta}} (\tilde\theta\Delta t - t)^2\,dt \right)^{\frac{1}{2}} \left(\int_{t_\theta}^{t_{\tilde\theta}} |u_{tt}(\cdot,t)|^2\,dt \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{3}}((1-2\theta)\,\Delta t)^{\frac{1}{2}} \left(\int_{t_\theta}^{t_{\tilde\theta}} |u_{tt}(\cdot,t)|^2\,dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| \frac{u^{(\tilde{\theta})} - \boldsymbol{u}^{(\theta)}}{(1-2\theta)\,\Delta t} - u_t^{(\theta)} \right\|^2 &\leq \frac{1}{3}\,(1-2\theta)\,\Delta t \int_\Omega \int_{t_\theta}^{t_{\tilde{\theta}}} |u_{tt}(\cdot,t)|^2 \, dt\, dA \\
&= \frac{1}{3}\,(1-2\theta)\,\Delta t \int_{t_\theta}^{t_{\tilde{\theta}}} \int_\Omega |u_{tt}(\cdot,t)|^2 \, dA\, dt \\
&\leq \frac{1}{3}((1-2\theta)\,\Delta t)^2 M^2 \\
&\leq C(\Delta t)^2. \qquad \blacksquare
\end{aligned}$$

Lemma C.20 *Assuming* $\|u_t\|_\infty < M$ *then*

$$\left\| u^{(\theta)} - u^{(0)} \right\|^2 \leq C(\Delta t)^2.$$

Proof of Lemma C.20: Make note that

$$\begin{aligned}
u^{(\theta)} - u^{(0)} &= \int_{t_\theta}^{t_0} 1 u_t \, dt \\
&\leq \left(\int_{t_0}^{t_\theta} 1 \, dt \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} u_t^2 \, dt \right)^{\frac{1}{2}} \\
&= \theta^{\frac{1}{2}} \Delta t^{\frac{1}{2}} \left(\int_{t_0}^{t_\theta} u_t^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| u^{(\theta)} - u^{(0)} \right\|^2 &= \int_\Omega \left| u^{(\theta)} - u^{(0)} \right|^2 \, dA \\
&\leq \theta \Delta t \int_\Omega \int_{t_0}^{t_\theta} u_t^2 \, dt\, dA \\
&= \theta \Delta t \int_{t_0}^{t_\theta} \int_\Omega u_t^2 \, dA\, dt \\
&\leq \theta \Delta t M^2 \int_{t_0}^{t_\theta} dt \\
&= \theta^2 M^2 \Delta t^2. \qquad \blacksquare
\end{aligned}$$

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